THE
AMERICAN
MATHEMATICAL MONTHLY.

A MONTHLY JOURNAL DEVOTED TO PURE MATHEMATICS.
PUBLISHED UNDER THE AUSPICES OF
THE UNIVERSITY OF CHICAGO.

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VOLUME XIII. JANUARY - DECEMBER, 1906.

PUBLISHED AT DRURY COLLEGE,
SPRINGFIELD, MO.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. JANUARY, 1906. No. 1.

ON THE IDENTICAL RELATIONS BETWEEN THE DETERMINANTS OF AN ARRAY.

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The object of this paper is a systematic account of the subject designated by Muir "vanishing aggregates of determinant products.* It will be seen that they are in general Laplacian expansions of vanishing determinants, which do not vanish when regarded as functions of the minor determinants. They include as a special case the fundamental determinant identity, the vanishing determinant with two identical rows or columns. This case (historically Vandermonde's identity†) underlies all the others. The first example of a more complicated kind is due to Bezout‡, who gave among the determinants of the 6×3 array the relation: $(123)(456)-(124)(356)+(134)(256)-(234)(156)=0$. Desnanot§ gave a number of new forms and the general method of deriving new from the old, called now the method of the extensional. More recently Cayley‖, Sylvester,¶ and Muir° have considered the general forms, while Kronecker,′ Runge″, and others have studied the special forms relating to an axisymmetric determinant.

* *The Theory of Determinants*, p. 52.
† Ib., p. 22.
‡ Ib., p. 52.
§ Muir, p. 143.
‖ *Coll. Math. Papers*, 1. 55.
¶ *Phil. Mag.*, 1850.
° *Proc. R. S. E.*, 1891, p. 73.
′ Sitzungsb, *d. k. Akad. d. Wiss.*, 1882.
″ Crelle, 93–319.

We consider the rectangular array whose elements are

$$a_{ik} \quad (i=1, 2, \ldots\ldots, m;\ k=1, 2, \ldots\ldots, n;\ m>n).$$

The transformation

$$a'_{ij}=\Sigma_k\, \lambda_{jk}\, a_{ik} \qquad |\,\lambda_{jk}\,| =1 \quad (j=1, \ldots\ldots, n)$$

leaves all determinants of n rows unaltered, while introducing n^2-1 arbitrary quantities among the elements. There are mn elements and $\binom{m}{n}$ determinants, hence at least $\binom{m}{n}-(mn-n^2+1)$ relations exist between the determinants which are identities in the elements. No more independent relations exist; for we can assign to $mn-n^2+1$ determinants arbitrary values as follows: The determinant formed by prefixing one column of the set to the array of any $n+1$ rows vanishes identically. When expanded by elements of the prefixed columns and their minors. This gives the fundamental determinant identities

$$\sum_{i=1}^{n+1} a_{ik}\,(a_{11}\; a_{22} \ldots\ldots\ldots a_{i-1.\, i-1}\; a_{i+1,\, i} \ldots\ldots\ldots a_{n+1,\, n}) \equiv 0,$$

one for each $k=1, 2, \ldots\ldots, n$. There are in all $n\binom{m}{n+1}$ such identities. They are lineo-linear in the elements and determinants of order n, homogeneous in elements, determinants, rows, and columns, and contain $n+1$ terms. We shall call them the *lineo-linear relations.*

Postulating that not all the determinants vanish, suppose that $(1, 2, 3, \ldots\ldots, n) \neq 0$. In this determinant give arbitrary values to n elements of $n-1$ columns and $n-1$ elements of the nth column provided that the minor of the remaining element be not zero. This last element can then be chosen uniquely so as to give $(1, 2, 3, \ldots\ldots, n)$ its assigned value. The lineo-linear relations may now be written $a_{n+j,\, k}\,(1, 2, \ldots\ldots, n) = -a_{n,\, k}\,(1, 2, \ldots\ldots, n-1, n+j) - \ldots\ldots = 0$, where j specifies any new row. There are $n(m-n)$ such relations and the determinant on the left and the elements on the right are assigned. Giving assigned values to the $n(m-n)$ determinants on the right we have unique values for the remaining $n(m-n)$ elements. They are finite for $(1, 2, \ldots\ldots, n) \neq 0$. All the elements and hence all the determinants are now assigned. The number of independent relations being at least, and not more than, is exactly

$$I=\binom{m}{n}-(mn-n^2+1).$$

In general however an indefinite number of relations exists, which may be reduced by definition and classification to a finite nnmber R, all of which must be satisfied. Except for some special cases, $R>I$ and a *plexus* exists. That is, in any special case some I of the R being satisfied the others follow by virtue of

relations among the R which are identities in the determinants. We proceed to the classification.

There exists no relation which is not either itself homogeneous in the elements of a certain set of rows, and of a certain set of columns, or obtainable by the composition of relations having these properties. The most general relation can be regarded as linear in the determinants of some set and as having for coefficients rational integral functions of the elements. Let $\Sigma C \triangle \equiv 0$ be such a relation. Choose a determinant containing a set of columns $(i, j, k, \ldots\ldots)$. Multiply these columns in the array by $s_i, s_j, s_k, \ldots\ldots$ The result must vanish for every s. Hence equating to zero the coefficients of the various powers and products of the s's we obtain a set of identities

$$\Sigma C_c \triangle_c = 0$$

homogeneous in a certain set of columns. A similar process in the rows gives a set of identities $\Sigma C_{cr} \triangle_{cr} = 0$, when each term contains the same set of rows and columns. Without loss of generality we may set $m \geq n$, where there are m rows and n columns. The original identity depends on and is composed of these. Since a single determinant cannot vanish identically the coefficient of each determinant must contain an element of some row not represented in this determinant but in others.

First consider possible relations between the $n+1$ determinants which enter a single set of $n+1$ rows. These can be obtained from a vanishing determinant whose jth row is

$$A_{j_1} a_{j_1} + A_{j_2} a_{j_2} + \ldots\ldots A_{j_n} a_{j_n}, \quad a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n}.$$

Consider the special case $a_{1k}=1$, $a_{2k}=a^k$, $a_{3k}=b^k$, ..., $a_{n+1,k}=z^k$, and put first $a=1$. We have

$$(A_{11}+A_{12}+\ldots\ldots)[(1, b, c^2, \ldots, z^n)] = (A_{21}+A_{22}+\ldots\ldots)[(1, b, c^2, \ldots, z^n)]$$

or $\Sigma A_{1k} = \Sigma A_{2k}$, for $(1, b, c^2, \ldots, z^n) \neq 0$. So in general $\Sigma A_{jk} = \Sigma A_{1k}$.

Returning to the general form, the A_{jk} each lack a kth column of being homogeneous in columns and a jth row of being homogeneous in rows. Hence, if in $\Sigma A_{1k} = \Sigma A_{2k}$ we place all the elements of the 1st column$=0$ we obtain $A_{11} = A_{21}$, and similarly $A_{jk} = A_{1k}$, and the relation reduces to a series of multiples of the lineo-linear relations.

Secondly we treat the case where determinants of more than $n+1$ rows enter.

Lemma. Out of m different things r sets of n things are taken, and for each of these sets of n, every set of $n+1$ including it is formed. If the number of these sets is s and $m-n \geq 2$ and *$m \geq 2$,* then *$ns > r$.*

Let the r sets be $a_{j_1}, a_{j_2}, a_{j_3}, \ldots, a_{j_n}$ $(j=1, 2, \ldots, n)$. Form the sets

$$(a_{j_1}, a_{j_2}, \ldots, a_{j_n})a_t \quad (j=1, 2, \ldots, r;\ t=1, 2, \ldots, n).$$

Of these nr have repeated elements leaving $(m-n)r$ which may in the complete system occur any number of times up to $n+1$. If T_f be the number occurring f times, $s=\Sigma \frac{T_f}{f}$ $(f=1, \ldots, n+1)$; $\Sigma T_f=(m-n)r$. Therefore

$$ns=\Sigma\frac{nT_f}{f}>\frac{n(m-n)r}{n+1}>r.$$

Now let determinants of more than one set of $n+1$ rows enter the relation. If s be the number of these sets from the ns lineo-linear relations belonging to the sets, multiply them by indeterminate coefficients, add and equate the coefficient of each determinant to the corresponding coefficient in the given relation. There are r linear homogeneous equations between ns indeterminates. They are independent; for, each involves a given set of rows, and as $ns>r$, rational integral functions of the elements can be found (in general in more than one way) for the indeterminates. This reduces the given relation to a sum of multiples of the lineo-linear relations.

We consider primarily then "vanishing aggregates of determinant products" of the second order. Those of the third order vanish only by being sums of products of determinants by such second order aggregates. The third order aggregates of which Monge* gave the first example, are important as relations used in elucidating the plexus between those of those of the second order.

A vanishing aggregate of determinant products of the second order, is then homogeneous in rows and columns, of order $2n$ in the elements. It contains at least three terms. For, the product of the determinants containing the rows $p, q, r, \ldots, w$ and $s, t, u, \ldots, z$, respectively, has a term

$$a_p\, b_q\, c_r, \ldots, l_w \,.\, a_s\, b_t\, c_u, \ldots, l_z$$

which can only be cancelled by a product from determinants such as $(p, t, r, \ldots, w)(s, q, u, \ldots, z)$ formed by simple interchange from first product, and this in turn has a term $a_t\, b_p\, c_r, \ldots, l_w \,.\, a_q\, b_s\, c_u, \ldots, l_z$ which does not occur in the first product.

We take a set of R such relations $(R \geq I)$ given by systematic enumeration and such that in every case these R are necessary and sufficient relations, but in any particular case (*i. e.* some one determinant not zero) a definite I of these relations are the independent ones. The set R may then be considered as fundamental.

First we have relations of $n+1$ terms given by the expansion of the identically vanishing determinant

*Muir, p. 68.

$$\begin{vmatrix}
a_1 & b_1 & c_1 \cdots\cdots l_1, & 0 & 0 & 0 \cdots\cdots 0 \\
a_2 & b_2 & c_2 \cdots\cdots l_2, & 0 & 0 & 0 \cdots\cdots 0 \\
\cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots & \cdots\cdots\cdots \\
a_{n-1} & b_{n-1} & c_{n-1} \cdots\cdots l_{n-1}, & 0 & 0 & 0 \cdots\cdots 0 \\
a_n & b_n & c_n \cdots\cdots l_n, & a_n & b_n & c_n \cdots\cdots l_n \\
a_{n+1} & b_{n+1} & c_{n+1} \cdots\cdots l_{n+1}, & a_{n+1} & b_{n+1} & c_{n+1} \cdots\cdots l_{n+1} \\
\cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots & \cdots\cdots\cdots \\
a_{2n} & b_{2n} & c_{2n} \cdots\cdots l_{2n}, & a_{2n} & b_{2n} & c_{2n} \cdots\cdots l_{2n}
\end{vmatrix}$$

As the choice of rows and of rows set opposite zeros is alone material we may denote this by

$$\frac{0,\ 0,\ 0, \ldots, 0,\quad n \mid n+1,\ n+2, \ldots, 2n}{1,\ 2,\ 3, \ldots, n-1,\ n \mid n+1,\ n+2, \ldots, 2n}$$

There are $\binom{m}{2n}\binom{2n}{n-1}$ of these, except for $n=2$ where the relation remains unchanged whichever row is placed opposite the zero. For $n=2$ the number is $\binom{m}{4}$.

This class of relations differs from the other classes to be discussed in important points. Since all the rows are different each determinant is multiplied by the same complementary determinant in each of the $\binom{2n}{n-1}$ relations formed from a given $2n$ rows. We may refer to this class as the *complementary class.*

Next we have relations of r terms $2<r<n+1$, given by the expansion of a similar vanishing determinant whose symbol is

$$\frac{0,\ 0, \ldots, 0,\quad n \mid n+1,\ n+2, \ldots, n+r-1,\ 1,\ 2, \ldots, n-r+1}{1,\ 2, \ldots, n-1,\ n \mid n+1,\ n+2, \ldots, n+r-1,\ 1,\ 2, \ldots, n-r+1}$$

The number is

$$\binom{n}{n+r-1}\binom{n+r-1}{n-r+1}\binom{2r-2}{r-2},$$

except for $r=3$ when the last factor is replaced by 1. The total for $r=3, 4, \ldots, n+1$ is

$$R=\binom{m}{n-1}\left[\binom{m}{n+1}-\frac{(m-n+1)(m-n)(mn-n^2-m+5)}{8}\right].$$

For $r<n+1$ the relations are "extensionals" of the relations of the complementary class for $m'=2n'$, where m' is the number of rows not used as "extenders." For example, $\frac{0\ 0\ 0\ 4 \mid 5\ 6\ 7\ 1}{1\ 2\ 3\ 4 \mid 5\ 6\ 7\ 1}$ may be regarded as derived from $\frac{0\ 0\ 4 \mid 5\ 6\ 7}{2\ 3\ 4 \mid 5\ 6\ 7}$ by using 1 as an "extender."

If there is an identical relation between complementary relations involving the same set of rows of the form $R_1+R_2+R_3=\ldots\ldots=0$, this relation may be extended by introducing any set of new rows into all the determinants entering. For example, the relation of complementary type of the 3×6 array are connected (among other connections) by

$$\frac{0\ 0\ 1 \mid 4\ 5\ 6}{1\ 2\ 3 \mid 4\ 5\ 6}+\frac{0\ 0\ 1 \mid 2\ 5\ 6}{3\ 4\ 1 \mid 2\ 5\ 6}+\frac{0\ 0\ 1 \mid 2\ 3\ 4}{5\ 6\ 1 \mid 2\ 3\ 4}\equiv 0,$$

which vanishes in the determinants. The relation

$$\frac{0\ 0\ 0 \ldots\ldots 0\ 0\ 3 \mid 4\ 5\ 6\ a\ b\ c \ldots\ldots}{a\ b\ c \ldots\ldots 1\ 2\ 3 \mid 4\ 5\ 6\ a\ b\ c \ldots\ldots}+\frac{0\ 0\ 0 \ldots\ldots 0\ 0\ 1 \mid 2\ 5\ 6\ a\ b\ c \ldots\ldots}{a\ b\ c \ldots\ldots 3\ 4\ 1 \mid 2\ 5\ 6\ a\ b\ c \ldots\ldots}$$

$$+\frac{0\ 0\ 0 \ldots\ldots 0\ 0\ 1 \mid 2\ 3\ 4\ a\ b\ c \ldots\ldots}{a\ b\ c \ldots\ldots 5\ 6\ 1 \mid 2\ 3\ 4\ a\ b\ c \ldots\ldots}\equiv 0$$

also holds, when $a\ b\ c \ldots\ldots$ is any set of rows. For, the vanishing of the simple relation depends on interchanges of rows, which interchanges are not affected by the extension.

Given any relation of complementary type

$$\frac{0\ 0\ 0 \ldots\ldots (r-1) \mid r,\ r+1,\ \ldots,\ 2(r-1)}{1\ 2\ 3 \ldots\ldots (r-1) \mid r,\ r+1,\ \ldots,\ 2(r-1)},$$

involving r terms, we obtain by extension $\binom{m-2r+2}{n-r+1}$ new relations. There are $\binom{m}{2(r-1)}\binom{2(r-1)}{r-2}$ such complementary relations, so that

$$\binom{n-r+1}{m-2r+2}\binom{m}{2(r-1)}\binom{2(r-1)}{r-2}=\binom{m}{n+r-1}\binom{n+r-1}{n-r+1}\binom{2r-2}{r-2}$$

is the number of relations of r terms in general.

The number of relations involving a given determinant is

$$K=\sum_{r=4}^{r=n+1}\left[2(r-1)\binom{m-n}{r-1}\binom{n}{r-1}\right]+\binom{m-n}{2}\binom{n}{2}.$$

For three-termed relations, we have the $(n-2)$nd extensions of the single relation $\frac{0\ b \mid c\ d}{a\ b \mid c\ d}$. $\binom{m-n}{2}$ is a factor of the number for the two rows in the given determinant, and $\binom{n}{2}$ a factor for choice of two rows not in the extenders. For $r>3$ we have a factor for the order of arrangement of $\frac{0\ 0 \ldots\ldots c \mid d\ e \ldots\ldots f}{a\ b \ldots\ldots c \mid d\ e \ldots\ldots f}$. If

a, b, ..., c belong to the given determinant, $(r-2)$ or none of these rows occur opposite the zeros. This gives $2(r-1)$ as the factor, and K as the total.

The relations of more than three terms involving a given determinant k are not independent but are connected with one another and those of their own type not involving k by linear relations whose coefficients are ± 1.

Take first the case $m=2n$ and the relations of $n+1$ terms. Write the $2n$ column numbers as

$$1,\ 2,\ 3,\ \ldots\ldots,\ n,\ \bar{n},\ (\overline{n-1}),\ \ldots\ldots,\ \bar{3},\ \bar{2},\ \bar{1}.$$

The relation

$$\left.\begin{array}{l} 0,\ 0,\ 0,\ \ldots\ldots\ldots\ldots,\ 0,\ n \\ 1,\ 2,\ 3,\ \ldots\ldots,\ (n-1),\ n \end{array}\right| \begin{array}{l} \bar{n},\ (\overline{n-1}),\ \ldots\ldots,\ \bar{2},\ \bar{1} \\ \bar{n},\ (\overline{n-1}),\ \ldots\ldots,\ \bar{2},\ \bar{1} \end{array}$$

contains $k=(1,\ 2,\ \ldots\ldots,\ n)$. So does each relation arising out of this by any interchanges of $1,\ 2,\ \ldots\ldots,\ n$; if $\bar{n},\ (\overline{n-1}),\ \ldots\ldots,\ \bar{2},\ \bar{1}$ are interchanged also, r and $\bar{r}$ going together, the leading terms have all the same sign. In general these interchanges do not alter the signs. The corresponding relation

$$\left.\begin{array}{l} 0,\qquad 0,\ \ldots\ldots\ldots\ldots,\ 0,\ 1 \\ (\overline{n-1}),\ (\overline{n-2}),\ \ldots\ldots,\ \bar{1},\ 1 \end{array}\right| \begin{array}{l} 2,\ 3,\ \ldots\ldots,\ n,\ \bar{n} \\ 2,\ 3,\ \ldots\ldots,\ n.\ \bar{n} \end{array}$$

and all the results of interchanges also contain k. If $n>2$ none of this set falls in the last mentioned. Add all these relations; the result is not an identity in the relations but vanishes in the determinants. For example, the products $[1,\ 2,\ \ldots\ldots,\ (n-1),\ \bar{a}][n,\ \bar{n},\ \ldots\ldots,\ (\overline{a+1}),\ (\overline{a-1}),\ \ldots\ldots,\ \bar{1}]$ and $[(\overline{n-1}),\ \ldots\ldots,\ (\overline{a+1}),\ \bar{a},\ (\overline{a-1}),\ \ldots\ldots,\ \bar{1},\ 1][2,\ 3,\ \ldots\ldots,\ n,\ \bar{n}]$ occur in the two sets. By the interchange $(1n)$ followed by $(1a)$, the latter becomes

$$[(\overline{n-1}),\ \ldots\ldots,\ (\overline{a+1}),\ \bar{1},\ (\overline{a-1}),\ \ldots\ldots,\ \bar{2},\ \bar{n}]$$
$$[2,\ 3,\ \ldots\ldots,\ (a-1),\ 1,\ (a+1),\ \ldots\ldots,\ (n-1),\ a,\ \bar{a}]$$

which is identical, except as to sign, with the first product. By suitable interchanges this pair of terms becomes other pairs and exhausts the two sets together. Hence we have an identity which may be written $\Sigma R(k)=0$ $(m=2n)$. A single interchange of a barred and unbarred column changes all the $R(k)$'s but two to $R(\bar{k})$'s, the exception being the two free from the interchanged columns. This interchange can be chosen so that for any one of the first set we have any one of the second left unchanged, and by subtraction we have for every pair of $R(k)$'s:

$$R_1(k)+R_2(k)+\Sigma R(\bar{k})=0 \quad (m=2n).$$

This relation can now be extended by any set of columns. As no such relations hold for the 2×4 array the extended relations hold only down to R_4.

The reduction of the $R(\bar{k})$ to dependence on the $R(k)$ can be accomplished as follows. Let

$$R(\bar{k}) = \frac{0\ 0\ 0\ 0 \ldots\ldots z \mid \alpha\ \beta\ \gamma \ldots\ldots \varsigma}{a\ b\ c\ d \ldots\ldots z \mid \alpha\ \beta\ \gamma \ldots\ldots \varsigma},\ k = (1,\ 2,\ \ldots,\ n).$$

Form the $3n$ row determinant whose upper right $2n \times 2n$ places are occupied by $R(\bar{k})$ and lower left $n \times n$ by k and fill in zeros below the zeros of $R(\bar{k})$ letters below the letters. Repeat k twice above itself. The symbol of it may be taken as

$$\begin{array}{l|l|l} 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ z & \alpha\ \beta\ \gamma\ \delta \ldots\ldots \varsigma \\ 1\ 2 \ldots\ldots n & a\ b\ c \ldots\ldots y\ z & \alpha\ \beta\ \gamma\ \delta \ldots\ldots \varsigma \\ 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ z & \alpha\ \beta\ \gamma\ \delta \ldots\ldots \varsigma \end{array}$$

Expanding by lower n rows, we get $D = k\,R(\bar{k}) + \Sigma\,a\,R$, when every R has at least one column which occurs in k. This determinant vanishes identically in the elements but not necessarily in the determinants. Expanding by the middle n rows, we see that only the determinants $[a\,b\,c \ldots\ldots y\,X]$ have non-vanishing coefficients. If X comes from the columns to the right the coefficient is 0 for n odd, and $2\,R(k)$ for n even. If X comes from k the coefficient contains $(n-1)$ columns of k if it does not vanish, and is expressible as $\Sigma\,R'_{n-1}$. We may then write

$$k\,R(\bar{k}) \equiv \Sigma\,a\,R_1 + \Sigma\,b\,R_{n-1} + \Sigma\,c\,R(k)$$

when every R_1 contains one column of k not opposite a zero. R'_{n-1} has $(n-1)$ columns of k in some position. The result is an identity in the determinants.

Take one of the R_1 and form a new $3n$ rowed determinant

$$\begin{array}{l|l|l} 1\ 2\ 3 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots\ldots 1 & \alpha\ \beta\ \gamma \ldots\ldots \varsigma \\ 1\ 2\ 3 \ldots\ldots n & a\ b\ c \ldots\ldots y\ 1 & \alpha\ \beta\ \gamma \ldots\ldots \varsigma \\ 1\ 2\ 3 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ 0 & \alpha\ \beta\ \gamma \ldots\ldots \varsigma \end{array}$$

Expanding by middle rows as before, we get $D \equiv \Sigma\,a\,R(k) + \Sigma\,b\,R'_n$. By lower rows, we reach

$$k\,R(\bar{k}) \equiv \Sigma\,a\,R_2 + \Sigma\,b\,R'_n + \Sigma\,c\,R(k).$$

Finally we reach R_{n-1}, and, by working on R'_{n-1} and on R'_n, we reduce them to R_{n-1}, *i. e.*, to relations involving $n-1$ rows of k not opposite the zero.

We then apply

$$\begin{array}{l|l|l} 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ 1 & 2\ 3 \ldots\ldots (n-1)\ \ \varepsilon\ \varsigma \\ 1\ 2 \ldots\ldots n & a\ b\ c \ldots\ldots y\ 1 & 2\ 3 \ldots\ldots (n-1)\ \ \varepsilon\ \varsigma \\ 1\ 2 \ldots\ldots n & 0\ 0\ 0 \ldots\ldots 0\ 0 & 0\ 0 \ldots\ldots\ldots 0\ \ \ \ \ \ \ \varepsilon\ \varsigma \end{array}$$

The $3n$ rowed determinant vanishes in the elements; for, subtraction of the upper n rows from middle leaves only $(n-1)$ columns. This time $D = \Sigma\,a\,R(k) + \Sigma\,b\,R_{n-1}$, when the symbols giving rise to R_{n-1} contain only two columns foreign to k and either vanishes identically or contains k. Expanding by lower rows, we get $D \equiv k\,R(\bar{k}) - \Sigma\,a\,R(k)$. Hence each $R(\bar{k})$ is reduced by a chain to depend on $R(k)$.

Now assume that all the $R_3(k)$ and one $R_r(k)$ in each set of columns vanish. The number of relations assumed satisfied is I, the total number of independent relations. Apply a series of reducing operations to

$$R_r(k)+R'_r(k)=\Sigma\, R''_r(k).$$

The $R''_r(k)$ have already n columns of k, and one more is introduced not opposite the zero. The result is an extensional with one more column among the extenders, *i. e.*, is an $R''_{r-1}(\bar{k})$. Hence the $R'_r(k)$ do not reappear in the chain. In treating the $R(\bar{k})$ it may happen that an extender is replaced on one side and and $R_r(\bar{k})$ lead to $R_{r+1}(\bar{k})$ or even to an $R_{r+1}(k)$, but these are reducible again, and eventually reduce to $R_3(k)$.

We finally have the theorem: *If* $k\neq 0$, *every relation is satisfied if every* $R_3(k)=0$, *and one* $R_r(k)=0$ *for every* r *and every set of columns.*

The number of independent determinants ot order n in an $m\times n$ array is thus in general $D=n(m-n)+1$.

In the particular case of zero values, however, either all vanish or there is an $(n-1)\times n$ array in which not all the minors of order $(n-1)$ vanish. Hence Cayley's theorem holds: $D_0=m-n+1$. If the $m-n+1$ determinants with $n-1$ columns of this array in common vanish, all others vanish. A general proof follows. A determinant not containing columns 1, 2, ..., $n-1$ has at least two columns not in this set. Hence the relation

$$\frac{0\ 0 \ldots\ldots 0\ \alpha \mid \beta\ \gamma \ldots s\ M}{1\ 2 \ldots n-1\ \alpha \mid \beta\ \gamma \ldots s\ M}=0,$$

gives $(\alpha\,\beta\,\gamma \ldots s)(1\ 2\ 3\ (n-1)M)=\Sigma(1\ 2\ 3 \ldots n-1\ \alpha)(M\,\beta\,\gamma \ldots s)$ if M be an arbitrary column and every minor of order $n-1$ from (1, 2, ..., $n-1$) be not zero, $(1, 2, \ldots, n-1, M)\neq 0$. Then from $(1, 2, 3, \ldots, n-1, \alpha)=0$, α any of $m-n+1$ columns, follows $(\alpha\ \beta\ \gamma \ldots s)=0$. As in the general case $m-n\geq 2$.

There are also relations between determinants of different orders taken from an array. The vanishing determinant

$$\begin{array}{r} \\ \\ \\ \\ r-1 \text{ rows}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \end{array}
\begin{vmatrix}
a_1 & b_1 \ldots f_1, & a_1 & b_1 & c_1 \ldots l_1 \\
a_2 & b_2 \ldots f_2, & a_2 & b_2 & c_1 \ldots l_2 \\
\ldots & \ldots\ \ldots & \ldots & \ldots & \ldots\ \ldots \\
a_{n+1} & b_{n+1}\ f_{n+1}, & a_{n+1} & b_{n+1} & c_{n+1} \ldots l_{n+1} \\
0 & 0 \ldots 0 & a_i & b_i & c_i \ldots l_i \\
0 & 0 \ldots 0 & a_j & b_j & c_j \ldots l_j \\
\ldots & \ldots\ \ldots & \ldots & \ldots & \ldots\ \ldots \\
\underbrace{0 & 0 \ldots 0}_{r \text{ columns}} & a_k & b_k & c_k \ldots l_k
\end{vmatrix}$$

gives r types of identities between $(n\times n)$ and $(r\times r)$ determinants. If s of the

set $(i, j, \ldots, n+1)$, there are $\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} m \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} m-n-1 \\ r-s-1 \end{bmatrix}$ of the type s $s=[0, 1, \ldots, (r-1)]$. If the array have M rows and N columns the total number of type s is $\begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} N \\ r \end{bmatrix}\begin{bmatrix} M \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} M-n-1 \\ r-s-1 \end{bmatrix}$. These relations between $(n\times n)$ and $(r\times r)$ determinants, $n>r$, can be reduced to sums of products of determinants of $(n-r)^2$ elements and the relations between determinants of the same order r added to products of $(n-r)$th order determinants and identities between determinants of order r which vanish in the determinants.

SEVERAL FUNDAMENTAL THEOREMS IN GROUP THEORY.

By DR. G. A. MILLER.

The following theorems are readily deduced from known theorems. They appear to be of sufficient importance to be explicitly stated in view of their numerous elementary applications.

Theorem I. *If a group contains more than one cyclic subgroup of order* $k=p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_\lambda^{\alpha_\lambda}$, *where* $p_1, p_2, \ldots, p_\lambda$ *are distinct prime numbers such that* $p_1<p_2<\ldots<p_\lambda$, *then it contains at least* p_1 *such subgroups. If the number of these subgroups is exactly* p_1 *the value of* α_1 *exceeds unity.*

This theorem may readily be derived from the known theorems that the number of cyclic subgroups of order p^α, $\alpha>1$ and $p>2$, in a non-cyclic group of order p^m is always divisible by p, and that the total number of subgroups of order p^α in such a group is of the form $1+mp$. Suppose that a group G contains more than one cyclic subgroup of order k but that the number of these subgroups is less than p_1+1. Each of these subgroups is transformed into itself by every other one, since an operator of order p^β transforms things in multiples of p when it does not transform them into themselves.

From this it follows that two of these cyclic subgroups generate a group which is the direct product of its Sylow subgroup. Since these Sylow subgroups cannot all be cyclic it is proved that this direct product contains at least p_1 cyclic subgroups of order k, and hence G contains at least p_1 cyclic subgroups of this order. From the fact that a group of order p^m cannot contain exactly p cyclic subgroups of order p^α when $\alpha=1$, it follows directly that $\alpha_1>1$ whenever G contains exactly p_1 cyclic subgroups of order k.

As an interesting particular consequence of this theorem we have that there is no group which contains exactly two cyclic subgroups of the same order when this order is either odd or twice an odd number. Moreover, it is easy to construct groups containing exactly two cyclic subgroups of any arbitrary order which is divisible by 4. For instance, there is an infinite number of groups containing just two cyclic subgroups of order twelve, but there is not a single

group which contains just two cyclic subgroups of order six. Similar remarks evidently apply to the other prime numbers.

Theorem II. *If a group contains exactly p cyclic subgroups of order k, then it contains only one subgroup of order* p^{a_λ}, $\lambda>1$, *and hence its Sylow subgroups whose orders are divisible by* p_λ *are cyclic.*

If these Sylow subgroups were not cyclic G would contain at least p_λ subgroups of order $p_\lambda{}^{a_\lambda}$. This is impossible since each of these subgroups would transform into itself each of the p_1 cyclic subgroups of order k, and hence it would also transform each of the operators of such a subgroup, with a possible exception of those whose orders are divisible by p_λ, into itself. From this it follows directly that there would be more than p_1 cyclic subgroups of order k in G. In the same manner it may be observed that if the order of G is divisible by any prime which exceeds p_λ all its operators whose orders are powers of this prime are commutative with each operator in the p_1 cyclic subgroups of order k.

Theorem III. *A necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that the nth power of each of its operators is contained in every subgroup of index n, for every possible value of n.*

That this condition is sufficient follows directly from the fact that if G would contain $1+kp$ Sylow subgroups of order p^m all the operators which would transform such a subgroup into itself would constitute a subgroup of index $1+kp$. As the latter would not contain all the operators whose orders are powers of p it could not contain the $1+kp$ power of every operator of G.

That the condition is also necessary follows from the fact that every subgroup of order p^a which is contained in a group of order p^m is itself invariant in a subgroup of order p^{a+1} and hence it includes the pth power of all the operators of the latter subgroup. Similarly, the latter subgroup contains the pth power of all the operators of a subgroup of order p^{a+2} and hence the given subgroup of order p^a contains the p^2 power of all the operators in this subgroup of order p^{a+2}. This reasoning can clearly be continued until we arrive at the entire group of order p^m.

It is well known that another necessary and sufficient condition that G is the direct product of its Sylow subgroups is that we arrive at the identity by finding the group of cogredient isomorphisms of G, and then finding the group of cogredient isomorphisms of this group of cogredient isomorphisms, and then the third group of cogredient isomorphisms, etc. A group which is its own group of cogredient isomorphisms contains no invariant operator besides the identity, and vice versa.

DEPARTMENTS.

DISCUSSION.

THE EVALUATION OF $\int_0^{\pi} \frac{\sin mx}{x} dx.$

By S. A. COREY, Hiteman, Iowa.

The following solution of problem 203, Calculus, the evaluation of the definite integral $\int_0^{\pi} \frac{\sin mx}{x} dx$ (m an integer), involves several points of interest.

$$\int_0^{\pi} \frac{\sin mx}{x} dx = \int_0^{m\pi} \frac{\sin x}{x} dx.$$

Developing $\int \frac{\sin x}{x} dx$ by the formula,*

$$f(x)=f(0)+\frac{x}{r2}\left\{[f'(x)+f'(0)]+2\left[f'\left[\frac{x}{r}\right]+f'\left[\frac{2x}{r}\right]+f'\left[\frac{3x}{r}\right]+\cdots\cdots\right.\right.$$

$$\left.\left.+f'\left[\frac{r-1}{r}x\right]\right]\right\}-\frac{B_1 x^2}{r^2.2!}[f''(x)-f''(0)]+\frac{B_2 x^4}{r^4.4!}[f^{\text{iv}}(x)-f^{\text{iv}}(0)]$$

$$-\frac{B_3 x^6}{r^6.6!}[f^{\text{vi}}(x)-f^{\text{vi}}(0)]+\cdots\cdots+(-1)^n\frac{B_n x^{(2n)}}{r^{(2n)}.(2n)!}[f^{(2n)}(x)-f^{(2n)}(0)]+\cdots\cdots (1),$$

(B_1, B_2, B_3, ..., being Bernoulli's numbers), and taking $r=2m$, we get

$$\int\frac{\sin x}{x}dx=c+\frac{x}{(2m).2!}\left\{\left[\frac{\sin x}{x}+1\right]+2\left(\frac{\sin\left[\frac{x}{2m}\right]}{\frac{x}{2m}}+\frac{\sin\left[\frac{2x}{2m}\right]}{\frac{2x}{2m}}+\frac{\sin\left[\frac{3x}{2m}\right]}{\frac{3x}{2m}}\right.\right.$$

$$\left.\left.+\cdots\cdots+\frac{\sin\left[\frac{2m-1}{2m}x\right]}{\frac{(2m-1)x}{2m}}\right)\right\}-\frac{x^2}{6.(2m)^2.2!}\left[\frac{x\cos x-\sin x}{x^2}\right]$$

$$+\frac{x^2}{30.(2m)^4.4!}\left[\left(\frac{6-x^2}{x^3}\right)\cos x+\left(\frac{6-3x^2}{x^4}\right)\sin x\right]\cdots\cdots(2).$$

Annals of Mathematics, Vol. V, No. 4, July, 1904.

But as m is an integer, $\int_0^{m\pi}\frac{\sin x}{x}dx$ develops by means of (2) into

$$\tfrac{1}{4}\pi+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots\ldots-(-1)^m\frac{1}{(2m-1)}\right)\pm\pi\left(\frac{1}{6.2^2.m.2!}+\frac{(m\pi)^2-6}{30.2^4.m^3.4!}\right.$$

$$\left.+\frac{(m\pi)^4-20\,(m\pi)^2+120}{42.2^6.m^5.6!}+\ldots\ldots\ldots\right)\ldots\ldots\ldots(3),$$

according as m is odd or even.

For convenient use in numerical computation (3) may be put into the form

$$\int_0^{m\pi}\frac{\sin x}{x}dx=\tfrac{1}{4}\pi+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots\ldots-(-1)^m\frac{1}{(2m-1)}\right)$$

$$\pm\left(\frac{c_1}{m}-\frac{c_3}{m^3}+\frac{c_5}{m^5}-\ldots\ldots\ldots\right)\ldots\ldots\ldots(4),$$

where $c_1=.0682995$, $c_3=.0019567$, $c_5=.0001948$, approximately.

By means of (4) the values of the definite integral corresponding to a few values of m are readily found to be as follows:

For $m=1$, 1.851936–
$m=2$, 1.418158+
$m=3$, 1.674760–
$m=4$, 1.492164+
$m=5$, 1.633963–
$m=6$, 1.518036–
....

SOLUTIONS OF PROBLEMS.

ALGEBRA.

247. Proposed by PROFESSOR G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Find the sum, to n terms, of

$$1+\frac{n}{2}+\frac{n(n+2)}{2.4}+\frac{n(n+2)(n+4)}{2.4.6}+\ldots\ldots\ldots$$

I. Solution by the PROPOSER.

The series is the coefficient of x^{n-1} in $(1-x)^{-\frac{1}{2}n}(1-x)^{-1}$; *i. e.*, in $(1-x)^{-(\frac{1}{2}n+1)}$. Hence the required sum is

$$\frac{(n+2)(n+4)\ldots\ldots(3n-2)}{2.4\ldots\ldots(2n-2)}.$$

II. Solution by HENRY HEATON, Atlantic, Iowa.

The sum of the first two terms is $\frac{n+2}{2}$; of the first three, $\frac{(n+2)(n+4)}{2.4}$; of the first four, $\frac{(n+2)(n+4)(n+6)}{2.4.6}$; of the first r, $\frac{(n+2)(n+4)\ldots\ldots(n+2r-2)}{2.4\ldots\ldots(2r-2)}$. This may be rigorously demonstrated by induction. If $r=n$ the required sum is

$$\frac{(n+2)(n+4)\ldots\ldots\ldots(3n-2)}{2.4\ldots\ldots\ldots(2n-2)}.$$

Also solved by L. E. Newcomb, J. Scheffer, and G. B. M. Zerr.

248. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{6435}{2}\cdot\frac{161280^2}{929569}\left[1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\ldots\right]=\pi^{16}$.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

In books on higher algebra it is proved that

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots\ldots=\frac{(2^{2n}-1)B_{2n-1}}{2n!}\cdot\frac{\pi^{2n}}{2},$$

B_{2n-1} being* the nth of Bernoulli's numbers.

$$\therefore 1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots=\frac{(2^{16}-1)\,B_{15}}{16!}\cdot\frac{\pi^{16}}{2},$$

$$\therefore \pi^{16}=\frac{2\times16!}{(2^{16}-1)\,B_{15}}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\ldots\right).$$

Since $B_{15}=\frac{3617}{510}$, we find $\frac{2\times16!\times510}{(2^{16}-1)3617}$, which agrees with the coefficient given in the problem.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots\ldots\ldots=\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\ldots\ldots$$

$$\therefore \log\left(1-\frac{4\theta^2}{\pi^2}\right)+\log\left(1-\frac{4\theta^2}{3^2\pi^2}\right)+\log\left(1-\frac{4\theta^2}{5^2\pi^2}\right)+\ldots\ldots\ldots$$

$$=\log\left\{1-\left[\frac{\theta^2}{2!}-\frac{\theta^4}{4!}+\frac{\theta^6}{6!}-\frac{\theta^8}{8!}+\frac{\theta^{10}}{10!}-\frac{\theta^{12}}{12!}+\frac{\theta^{14}}{14!}-\frac{\theta^{16}}{16!}+\ldots\ldots\ldots\right]\right\}$$

$$=\log(1-A).$$

*Chrystal's *Algebra*, Vol. II, Chapter 30, §15.

$$\therefore -\frac{4\theta^2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)-\frac{4^2\theta^4}{2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots\right)-\ldots\ldots$$

$$-\frac{4^8\theta^{16}}{8\pi^{16}}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}\right)=-A-\tfrac{1}{2}A^2-\ldots\ldots-\tfrac{1}{8}A^8-\ldots\ldots$$

Equating like coefficients of θ^{16} we get

$$\frac{8192}{\pi^{16}}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right)=\frac{929569}{6435\times1260^2}.$$

$$\therefore \frac{6435}{2}\cdot\frac{161280^2}{929569}\left(1+\frac{1}{3^{16}}+\frac{1}{5^{16}}+\frac{1}{7^{16}}+\ldots\ldots\right)=\pi^{16}.$$

Also solved by F. Anderegg, and G. W. Greenwood.

249. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $x+y+z=5$, $x^2+y^2=z^2$, $x^3+y^3+z^3=8$.

Solution by M. R. BECK, Cleveland, Ohio.

$x+y+z=5$, or $x+y=5-z$.........(1), $x^2+y^2=z^2$.........(2),
$x^3+y^3+z^3=8$, or $(x+y)(x^2-xy+y^2)=8-z^3$.........(3).

From (1) and (2), we have $xy=\dfrac{25-10z}{2}$.........(4).

Substituting (1), (2), and (4) in (3), and solving, $z=\frac{47}{25}$. Substituting $z=\frac{47}{25}$ in (1) and (2) and solving, $x=\frac{39}{25}\mp\frac{7}{50}\sqrt{-34}$, and $y=\frac{39}{25}\pm\frac{7}{50}\sqrt{-34}$.

Also solved by Henry Heaton, A. H. Holmes, L. E. Newcomb, J. Scheffer, and G. B. M. Zerr.

AVERAGE AND PROBABILITY.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

What is the average length of all straight lines that can be drawn within a given triangle?

II. Solution bv HENRY HEATON, Atlantic, Iowa.

Let ABC be the triangle. Let x=length of one of the straight lines, and θ the angle it would make with the side AB if produced to meet it, θ being taken $>$ than the angle A and $<$ than $\pi-B$. At the distance $x\sin\theta$ from AB draw ED parallel to AB cutting AC in E and BC in D. Then the number of lines of length x making the angle θ with AB is equal to the number of points in the triangle DEC whose area is $\dfrac{\triangle\,(b\sin A-x\sin\theta)^2}{b^2\sin^2A}$. Similarly, the number of lines of length x making the angle θ with AC is equal to the number of points in the triangle whose area is $\dfrac{\triangle\,(c\sin A-x\sin\theta)^2}{c^2\sin^2A}$, and the number making same angle with the side BC is equal to the number of points in the triangle whose area is

$\frac{\triangle\ (c\sin B - x\sin\theta)^2}{c^2\sin^2 B}$. Then if all lines are equally distributed about the starting point the required average is

$$M_1 = \frac{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2 A} d\theta x dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2 A} d\theta x dx + \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B} d\theta x dx}{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2 A} d\theta\, dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2 A} d\theta dx + \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B} d\theta dx}$$

$$= \frac{\int_A^{\pi-B} \frac{b^2\sin^2 A}{\sin^2\theta} d\theta + \int_A^{\pi-C} \frac{c^2\sin^2 A}{\sin^2\theta} d\theta + \int_B^{\pi-C} \frac{c^2\sin^2 B}{\sin^2\theta} d\theta}{4\int_A^{\pi-B} \frac{b\sin A}{\sin\theta} d\theta + 4\int_A^{\pi-C} \frac{c\sin A}{\sin\theta} d\theta + 4\int_B^{\pi-C} \frac{c\sin B}{\sin\theta} d\theta}$$

$$= \tfrac{3}{4}abc \div ab \log\left(\frac{s}{s-c}\right) + ac\ \log\left(\frac{s}{s-b}\right) + bc \log\left(\frac{s}{s-a}\right).$$

If the lines are so distributed as to join every possible pair of points the required average is

$$M_2 = \frac{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2\theta} d\theta x^2 dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2\theta} d\theta x^2 dx + \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B} d\theta\, x^2 dx}{\int_A^{\pi-B}\int_0^{\frac{b\sin A}{\sin\theta}} \frac{(b\sin A - x\sin\theta)^2}{b^2\sin^2 A} d\theta x dx + \int_A^{\pi-C}\int_0^{\frac{c\sin A}{\sin\theta}} \frac{(c\sin A - x\sin\theta)^2}{c^2\sin^2 A} d\theta x dx + \int_B^{\pi-C}\int_0^{\frac{c\sin B}{\sin\theta}} \frac{(c\sin B - x\sin\theta)^2}{c^2\sin^2 B} d\theta x dx}$$

$$= \tfrac{2}{5} \frac{\left[\int_A^{\pi-B} \frac{b^3\sin^3 A}{\sin^3\theta} d\theta + \int_A^{\pi-C} \frac{c^3\sin^3 A}{\sin^3\theta} d\theta + \int_B^{\pi-C} \frac{c^3\sin^3 B}{\sin^3\theta} d\theta\right]}{\int_A^{\pi-B} \frac{b^2\sin^2 A}{\sin^2\theta} d\theta + \int_A^{\pi-C} \frac{c^2\sin^2 A}{\sin^2\theta} d\theta + \int_B^{\pi-C} \frac{c^2\sin^2 B}{\sin^2\theta} d\theta}$$

$$=\tfrac{1}{15}\left(a+b+c+(a+b)\frac{(a-b)^2}{c^2}+(a+c)\frac{(a-c)^2}{b^2}+(b+c)\frac{(b-c)^2}{a^2}\right)$$

$$+\tfrac{4}{15}\triangle^2\left[\frac{1}{c^3}\log\left(\frac{s}{s-c}\right)+\frac{1}{b^3}\log\left(\frac{s}{s-b}\right)+\frac{1}{a^3}\log\left(\frac{s}{s-a}\right)\right].$$

In the above free use is made of the well known formulas

$$\sin A=\frac{2\triangle}{bc},\quad \cos A=\frac{b^2+c^2-a^2}{bc},\quad \tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \text{etc.}$$

*If $c=b=a$, $M_1=\dfrac{a}{4\log 3}=.2276a$, and $M_2=\frac{1}{5}+\frac{4}{5}\dfrac{\triangle^2}{a^3}\log 3=\dfrac{a}{5}(1+\frac{3}{4}\log 3)$ $=.3638a$.

CALCULUS.

Note on Problem 207.—The proposer's value of $\frac{1}{4}\pi^2$ for the definite integral $\int_0^1\frac{Kd_\kappa}{1+\kappa}$ is wrong and should be $\frac{1}{8}\pi^2$. Mr. Corey's solution (p. 230) is correct except in the last line where the evaluation of $u-v=2\int_0^1\frac{\log(1+z)}{z}dz-\int_0^1\frac{\log(1+z^2)}{z}$ should read $2\frac{\pi^2}{12}-\frac{1}{2}\frac{\pi^2}{12}=\frac{\pi^2}{8}$. That this is true is easily shown by the following evaluation, due to Dr. Zerr, who has contributed two correct solutions of the problem.

$$u-v=2\int_0^1(1-\tfrac{1}{2}z+\tfrac{1}{3}z^2-\tfrac{1}{4}z^3+\ldots)dz-\int_0^1(z-\tfrac{1}{2}z^3+\tfrac{1}{3}z^5-\tfrac{1}{4}z^7+\ldots)dz$$

$$=2(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots)-\tfrac{1}{2}(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots)$$

$$=-\tfrac{3}{2}(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots-\frac{2}{1^2}-\frac{2}{3^2}-\frac{2}{5^2}-\ldots)\dagger=-\tfrac{3}{2}(\frac{\pi^2}{6}-\frac{\pi^2}{4})=\frac{\pi^2}{8}.$$

G.

210. Proposed by EDWIN L. RICH, Schenectady, New York.

Determine a polynomial, $f(x)$, entirely in x and of the seventh degree, so that $f(x)+1$ is divisible by $(x-1)^4$ and $f(x)-1$ by $(x+1)^4$. [Goursat-Hedrick, *A Course in Mathematical Analysis*, p. 32, Ex. 3.]

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*See Problem 35, p. 391, Williamson's *Integral Calculus*.

†See for example, Locke's *Treatise on Higher Trigonometry*, p. 99 and Ex. 1, p. 100.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

$$f(x)\equiv(x-1)^4(ax^3+bx^2+cx+d)-1\equiv(x+1)^4(a'x^3+b'x^2+c'x+d')+1.$$

Putting $-x$ for x and transposing we have

$$(x-1)^4(a'x^3-b'x^2+c'x-d')-1\equiv(x+1)^4(ax^3-bx^2+cx-d)+1.$$

These identities are consistent if $a=a'$, $b=-b'$, $c=c'$, $d=-d'$.

By equating coefficients of the same powers of x in either identity, and solving the resulting equations we get $a=\frac{5}{16}$, $b=\frac{5}{4}$, $c=\frac{29}{16}$, $d=1$.

$\therefore f(x)\equiv\frac{5}{16}x^7-\frac{21}{16}x^5+\frac{35}{16}x^3-\frac{35}{16}x.$

Also solved by R. D. Carmichael, A. H. Holmes, Henry Heaton, J. Scheffer, G. B. M. Zerr, and the Proposer.

211. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

*If $x=v^{1/(v-1)}$, what is the $f(x)$ such that $v=f(x)$?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $v=u+1$. Then $x^u=u+1$. Let $x=e^c$. $\therefore e^{cu}=u+1$. Let $cu=y$. $\therefore e^y=y/c+1$. $\therefore y=-c+ce^y$. If $y=a+b\varphi(y)$ we have by Lagrange's Theorem

$$y=a+b\varphi(a)+\frac{b^2}{2!}\cdot\frac{d}{da}[\varphi(a)]^2+\ldots+\frac{b^n}{n!}\cdot\frac{d^{n-1}}{da^{n-1}}[\varphi(a)]^n+\ldots \text{ etc.}$$

In this problem $\varphi(a)=e^a$.

$$\therefore y=-c+ce^{-c}+c^2e^{-2c}+\frac{c^3}{2!}e^{-3c}+\frac{c^4}{3!}e^{-4c}+\ldots+\text{ etc.}$$

$$\therefore u+1=v=f(x)=e^{-c}+ce^{-2c}+\frac{c^2}{2!}e^{-3c}+\frac{c^3}{3!}e^{-4c}+\ldots \qquad c=\log_e x,\ e^{-c}=1/x.$$

$$\therefore v=f(x)=1/x+\frac{\log_e x}{x^2}+\frac{(\log_e x)^2}{2!x^3}+\frac{(\log_e x)^3}{3!x^4}+\ldots+\frac{(\log_e x)^n}{n!x^{n+1}}+\ldots\text{ etc.}$$

$$\therefore v=f(x)=x^{(1-x)/x}.$$

GEOMETRY.

267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles in a plane is an isothermal system.

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*This problem should admit of interesting generalizations, say for $x=v^{f(v)}$ for certain classes of functions f. G.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let any given orthogonal system of circles be inverted with respect to a circle whose center is an intersection of two circles, one from each family, and neither of them a real circle. The result is a new orthogonal system containing two straight lines derived from the two circles, and each line is the locus of centers of the 'opposite' family of circles. Using these lines as axes of coördinates, the two circle families are $(x-a)^2+y^2=c^2$, $x^2+(y-b)^2=d^2$, in which, because the circles are orthogonal, $a^2+b^2=c^2+d^2$. Writing $c^2=a^2-k^2$ in the last equation gives $d^2=b^2+k^2$, and the circle families become $x^2+y^2-2ax+k^2=0$, $x^2+y^2-2by-k^2=0$.

The constants are now independent, but since any circle of one family is orthogonal to all of the other family it follows that a and b are the respective parameters. If now a and b are replaced by $k\coth 2v$ and $-k\cot 2u$, respectively, the system may be written $u+vi=\tan^{-1}\dfrac{x+yi}{k}$, which shows that it is isothermal. Thus the given system is also isothermal, since it may be obtained from this one by inversion. When k is 0 or ∞ the corresponding result is

$$u+vi=\frac{1}{x+yi},\text{ or } u+vi=x+yi.$$

274. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a straight line AB is placed between two intersecting straight lines MN and PQ and is made to revolve through all possible positions having A always in MN and B always in PQ, what is the locus of any point L in AB or AB produced?

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon. Ill.

We can choose coördinate axes so that the equations to the given lines are $y=rx$, $z=a$; $y=-rx$, $z=-a$. Let the coördinates of A, B, L be, respectively, (h, rh, a), $(k, -rk, -a)$, (x, y, z). Then

$$\frac{x-h}{x-k}=\frac{y-rh}{y+rk}=\frac{z-a}{z+a}=\frac{AL}{BL}=m,\text{ say.}$$

$\therefore h-mk=x(1-m)$..............(1), $r(h+mk)=y(1-m)$................(2), $z(1-m)=a(1+m)$................(3).

Hence the locus lies in a plane parallel to $z=0$, or to the given lines as is otherwise evident. Also $AB^2=l^2=(h-k)^2+r^2(h+k)^2+4a^2$................(4). Eliminating h, k between (1), (2), (4) we have an ellipse for the required locus, its equation being

$$(1-m)^2\{[y(m-1)+rx(m+1)]^2+[y(m+1)+rx(m-1)]^2r^2\}=4m^2r^2(l^2-4a^2).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the intersection of MN, PQ. $OA=a$, $OB=b$. Draw LD parallel to OB, and let D be in MN. Let $OD=u$, $DL=v$, $AB=c$, $AL=d$, $\angle AOB=\beta$. Then $a^2+b^2-2ab\cos\beta=c^2$; $c:d=a:a\pm u$; $c:d=b:v$.

$$\text{Hence } a=\pm\frac{cu}{d-c},\ b=\frac{cv}{d},\text{ and }\frac{c^2u^2}{(d-c)^2}+\frac{c^2v^2}{d^2}\mp\frac{2c^2uv\cos\beta}{d(d-c)}=c^2.$$

$$\therefore\ \frac{u^2}{(d-c)^2}+\frac{v^2}{d^2}\mp\frac{2uv\cos\beta}{d(d-c)}=1.$$

$\therefore$ The locus is an ellipse.

III. Solution by A. H. HOLMES, Brunswick, Maine.

Suppose the straight lines MN and PQ intersect each other at right angles at O, and AB placed between them: A on MN and B on PQ, and L a point in AB. Draw LO. Put $AL=b$, $BL=a$, and $LO=r$, and $LAO=\phi$, $AOL=\theta$. Then $b\sin\phi=r\sin\theta$, and $a\cos\phi=r\cos\theta$.

$$\therefore \sin^2\phi=\frac{r^2}{b^2}\sin^2\theta,\text{ and }\cos^2\phi=\frac{r^2}{a^2}\cos^2\theta.\quad \therefore r=\frac{ab}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}}.$$

Therefore the locus of point L is an ellipse whose semi-major axis is BL and whose semi-minor axis is AL. When MN and PQ intersect obliquely at angle ψ the semi-minor axis would be $\dfrac{ab\sin\psi}{\sqrt{(a^2-b^2\cos^2\psi)}}$.

Also solved by R. D. Carmichael, and J. Scheffer.

275. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

An hyperbola is drawn touching the axes of an ellipse, and the asymptotes of the hyperbola touch the ellipse. Prove that the center of the hyperbola lies on one of the equal conjugate diameters of the ellipse.

Solution by the PROPOSER.

Let (x', y') be the intersection of the tangents to the ellipse $a^2y^2+b^2x^2-a^2b^2=0$ .... (1); then these tangents being the asymptotes of the hyperbola, (x', y') is the center of the hyperbola. The equation to the tangents to (1) from (x', y') is

$$(a^2y^2+b^2x^2-a^2b^2)(a^2y'^2+b^2x'^2-a^2b^2)=(a^2y'y+b^2x'x-a^2b^2)^2\text{..........(2)},$$

$$\text{or, } (y'^2-b^2)x^2+(x'^2-a^2)y^2-2x'y'xy+2b^2x'x+2a^2y'y-(a^2y'^2+b^2x'^2)=0\text{.....(3)}.$$

Now, the equation to the asymptotes of a conic differs from the equation to the conic by a constant only; then adding c to the left member of (3) we have the equation to the hyperbola.

If now $y=0$ in this equation to the hyperbola, we have

$$(y'^2-b^2)x^2+2x'x-(a^2y'^2+b^2x'^2)+c=0 \text{.........} (4),$$

and the condition that (4) has equal roots, or that the hyperbola touches the X-axis is given by $y'^2(a^2b^2-a^2y'^2-b^2x'^2)=c(b^2-y'^2)$ .... (5); and, in a similar way that the curve touches the Y-axis, $x'^2(a^2b^2-a^2y'^2-b^2x'^2)=c(a^2-x'^2)$ ....(6). (5)$\div$(6) gives after reduction, $a^2y'^2-b^2x'^2=0$..........(7), showing that (x', y') is on an equi-conjugate axis of the ellipse.

Also solved by G. W. Greenwood, A. H. Holmes, W. W. Landis, J. Scheffer, and G. B. M. Zerr.

GROUP THEORY.

12. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics, The University of Pennsylvania.

Given $U_1=\alpha'$, $V_1=\beta'$, and the recursion formulae $U_y=\alpha' V_{y-1}+\alpha'' U_{y-1}$, $V_y=\beta' V_{y-1}+\beta'' U_{y-1}$. Find expressions for U_y, V_y in terms of the coefficients α', α'', β', β''.

Solution by PROFESSOR JAMES BYRNIE SHAW, The James Milliken University, Decatur, Ill.

By eliminating V we find that U_n is the coefficient of x^n in the expansion of

$$\frac{\alpha' x}{1-(\alpha''+\beta')x+(\alpha''\beta'-\alpha'\beta'')x^2}.$$

Likewise we find that V_n is the coefficient of x^n in the expansion of

$$\frac{1-\alpha'' x}{1-(\alpha''+\beta')x+(\alpha''\beta'-\alpha'\beta'')x^2}.$$

We may state the result as follows: Let $\cos\theta=\frac{1}{2}.\dfrac{\alpha''+\beta'}{T}$ where $T^2=\begin{vmatrix} \alpha'' & \alpha' \\ \beta'' & \beta' \end{vmatrix}$.

Then $V_n=\alpha'.T^{n-1}.\dfrac{\sin n\theta}{\sin\theta}$, and $V_n=T^n.\dfrac{\sin(n+1)\theta}{\sin\theta}-\alpha'' T^{n-1}\dfrac{\sin n\theta}{\sin\theta}$. These latter forms are easily verified by mathematical induction. The well-known formulae for $\dfrac{\sin n\theta}{\sin\theta}$ give U_n and V_n in terms of the coefficients directly, and free from irrationalities.

13. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The order of the linear homogeneous group in n letters is $(p^n-1)(p^n-p)$ ....(p^n-p^{n-1}). Two proofs are given in Burnside's ***Finite Groups***. Give other proofs.

Solution by the PROPOSER.

The linear homogeneous group is known to be equivalent to the group of isomorphisms of the abelian group $H_{p^n}=[P_1, P_2 \text{..........} P_n]$ of type [1 1 1..........],

order p^n. Let $h_{i_1}\ h_{i_2}$ .... represent the subgroups of H, of order p, and $J_{ik}=\begin{pmatrix}P_1 & P_2 & \dots\dots\dots P_n\\ h_{i_1} & h_{i_2} & \dots\dots\dots h_{i_n}\end{pmatrix}$ the isomorphism of H gotten by replacing P_j by any operation (order p) in h_{i_j} ($j=1\ 2$ .... m), say the new generators from the kth set of all of the possible sets which might be chosen from h_{i_1}, h_{i_2} .... h_{i_n}. The number of values of k is obviously equal to $\Phi(p)^n=(p-1)^n$. To determine the number of choices *of this set* of subgroups (number of values of i) suppose that a of a set of n generators have been selected. The remaining $n-a$ operations must be selected outside the subgroup H_{p^a} generated by the first a, and thus there remain

$$\frac{p^n-1}{p-1}-\frac{p^a-1}{p-1}=\frac{p^a(p^{n-a}-1)}{p-1}$$

subgroups h_{i_j} from which to select the remaining $n-a$. Thus the product of the number of values of k and the number of values of i is

$$h=(p-1)^n\prod_{a=0}^{n-1}\frac{p^a(p^{n-a}-1)}{p-1}=(p^n-1)(p^n-p)(p^n-p^2)\dots\dots\dots(p^n-p^{n-1})$$

which is the number of choices of new generators of H, or the order of its automorph.

MECHANICS.

186. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

A point P keeps at uniform distance from and moves with uniform angular velocity around a point Q which is in harmonic motion, making one revolution while Q swings to and fro. If P is in the line of the path of Q and on the same side of the center of that path with Q when Q is at the extremity of the path, what is the locus of P?

Solution by the PROPOSER.

Take the origin at the center of the path of Q, and let a=half the length of that path. Let $PQ=b$, and let θ=the angle of PQ with the path of Q at any time. Then, it is easily shown that $x=(a+b)\cos\theta$, $y=b\sin\theta$, the equations of an ellipse whose axes are $a+b$ and b.

Also solved by G. W. Greenwood, and G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

253. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Prove that $x^5+ax+b=0$ is solvable by radicals if $b=ma$, m being the negative of half the sum of any two roots of the original equation. Exhibit the solution.

254. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\dfrac{n^2}{(16n^2-1)^2}$ beginning with $n=1$.

255. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Let f be the binary cubic $a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_1^2+a_3x_2^3$, $\triangle=(f,f)_2$ the covariant, the second transvectant of f over itself, and $R=2[4(a_0a_2-a_1^2)\times(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2]=(\triangle,\triangle)_2$=the second transvectant of $\triangle$ over itself. Then if $\triangle_{\kappa\lambda}$ is the $\triangle$ covariant for the cubic pencil $\kappa f+\lambda Q$, Q being the first transvectant of f over $\triangle$ we have $\triangle_{\kappa\lambda}=(\kappa^2-\frac{1}{2}\lambda^2R)\triangle$.

CALCULUS.

900. Proposed by PROFESSOR B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Prove that, if the differential equation $cydx-(y+a+bx)dy-nx(xdy-ydx)=0$, be transformed into an equation between u and x by the substitution $u(y+a+bx+nx^2)=y(c+nx)$, then the variables are separable; and reduce the equation to the form $dv/\phi(v)=dx/\phi(x)$ by the further substitution $v=\alpha u+\beta$, α and β being suitably determined. *Euler.* [Forsyth's *Differential Equations*, p. 48, Ex. 4.]

DIOPHANTINE ANALYSIS.

132. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Disregarding the order of λ, μ, ν, how many sets of solutions has the congruence $\lambda+\mu+\nu\equiv 0 \pmod{p-1}$ (p prime)?

GEOMETRY.

280. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

On any diameter of a given ellipse is taken a point such that the tangents from it intercept on the tangent at one end of the diameter a length equal to the diameter; the ellipse being $a^2y^2+b^2x^2-a^2b^2=0$. Prove that the locus of the point is $\left(\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right)^2=\left(\dfrac{a^2+b^2}{a^2-b^2}\right)^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)$.

281. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

In the proposition in solid geometry "If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane of the lines," it is assumed that two intersecting lines have a common perpendicular. Prove it.

282. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The pedal lines of any two points on the circumcircle of a triangle concur in an angle equal to that subtended by the said points.

283. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The right angled intersection of the pedal lines of any diameter of the circumcircle lies on the "nine points circle" of the inscribed triangle.

MECHANICS.

990. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A curve in a vertical plane has a horizontal tangent along which a heavy particle moves without friction with uniform velocity v until it reaches T, the point of tangency, when by its own momentum alone it ascends the curve against the force of gravity. Find the equation of the curve such that the particle, after ascending for a time along the curve, will leave it and fall freely till it strikes at T.

NOTES AND NEWS.

Mr. E. B. Smith, instructor in mathematics at Purdue University, has resigned.

Mr. W. M. Persons, of the University of Wisconsin, has been appointed assistant professor of mathematics at Dartmouth College.

Professor J. E. Bonebright, of the Colorado Agricultural College, has been appointed professor of mathematics at Ottawa University, Ottawa, Kansas.

The following officers were elected at the Chicago meeting of the Central Association of Science and Mathematics Teachers: President, O. W. Caldwell, Charleston, Ill.; Treasurer, C. W. D. Parsons, Evanston, Ill.

Dr. S. T. Tamura, B. Sc., M. A. (Iowa), Ph. D. (Columbia), a native of Japan, has been appointed mathematician in the department of terrestrial magnetism of the Carnegie Institution, with which he has been connected as assistant for the past two years.

The Mathematical Section of the California Teachers Association elected the following officers at its recent meeting: President, Professor G. A. Miller; Vice President, Professor W. H. Baker, San Jose Normal School; Secretary, Mr. J. Fred Smith, Campbell High School.

Professor Alexander Ziwet, Chairman and Vice President of Section A, American Association for the Advancement of Science, took for the subject of his New Orleans vice presidential address "The Relation of Mechanics to Physics." His address is published in full in *Science* of January 12.

The Colorado Mathematics Society, having for its main object the improvement of the teaching of mathematics, was organized in Denver, Colorado, on December 2, 1905. Professor DeLong, of the University of Colorado, was elected president, and Mr. Smith, of the North Denver High School, secretary.

The Proceedings of the International Congress of Arts and Sciences, which held its meeting at St. Louis in September, 1904, are being published by Houghton, Mifflin Company. These will comprise eight volumes, the first of which will contain papers on philosophy and mathematics. The papers will be published as presented, except that those read in foreign languages will be translated into English.

A NATIONAL SOCIETY, PRO AND CON.

The report of the committee on "American Society of Teachers of Mathematics and the Natural Sciences" is published in the January number of *School Science and Mathematics.* This committee has drafted a constitution which it has submitted to the leading local mathematics and science teachers associations in the country for their approval. The objects of the American Society as set forth in Article II of the Constitution are, (1) "To improve the teaching of mathematics and science." (2) "To promote the interests of teachers of mathematics and the natural sciences." (3) "To foster the organization of such teachers in groups of various scopes." (4) "To support a journal or journals devoted to the aims of the society." It is proposed in the Constitution to hold one annual meeting in conjunction with the N. E. A.

A significant response to the circulation of the committee's report was the resolution of the Central Association at Chicago December 2, that "the proposed constitution does not meet the ideals of the Central Association of Science and Mathematics Teachers." This association passed resolutions favoring the organization of three coöperating general societies, eastern, western, and central, each holding its own general meetings within its own territory, the three uniting as a national body in one annual meeting.

ASSOCIATION OF TEACHERS OF MATHEMATICS IN INDIANA.

The Mathematical Section of the Indiana State Teachers' Association held its annual meeting as a section of the Association at Indianapolis, on December 27, about 250 teachers of mathematics being present. The following papers were read: (1) The High School's Portion of Higher Mathematics, by Professor David A. Rothrock, Indiana University; (2) Instructing vs. Teaching,

by Professor John C. Stone, Indiana State Normal; (3) In What Grades Should the Study of Algebra Begin? by Superintendent George L. Roberts, Muncie; (4) How Can High School Mathematics Better Prepare for Study of Science? by Leonard Young, Evansville High School. By a vote of the Section a committee of five, consisting of Professor D. A. Rothrock, Indiana University, Bloomington, Professor T. G. Alford, Purdue University, Lafayette, Professor W. P. Morgan, State Normal, Terre Haute, Principal D. R. Ellabarger, Richmond High School, and Mr. G. H. Mingle, Anderson High School, was appointed to formulate plans for an "Association of Teachers of Mathematics in Indiana." The committee was empowered to call a meeting and arrange a program for preliminary organization. The first meeting of the proposed Association will be held at Indianapolis, March 30, in connection with the meeting of the Southern Indiana Teachers' Association.

THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society held its twelfth annual meeting at Columbia University on December 28-29, simultaneously with the meetings of the American Physical Society, and the Astronomical and Astrophysical Society of America. Large attendance and extensive programs characterized the threefold gathering. A joint meeting of the Mathematical and Physical societies was held on Friday afternoon for the purpose of hearing Professor V. F. Bjerknes, of the University of Stockholm, who spoke on "Experimental Demonstration of Hydro-dynamic Action at a Distance."

The American Mathematical Society elected the following officers and members of the Council: Vice Presidents, Charlotte A. Scott and Irving Stringham; Secretary, F. N. Cole; Treasurer, W. S. Dennett; Librarian, D. E. Smith; Committee of Publication, F. N. Cole, Alexander Ziwet, D. E. Smith; Members of the Council, to serve until December, 1908, C. L. Bouton, L. E. Dickson, Edward Kasner, E. J. Townsend.

The following have been elected members of the Society: Mr. R. L. Börger, University of Missouri; Professor W. B. Cairns, Ursinus College; Mr. A. J. Champreux, University of California; Dr. Emily Coddington, New York, N. Y.; Dr. F. J. Dohmen, University of Texas; Dr. O. E. Glenn, Drury College; Mr. E. S .Haynes, University of Missouri; Professor J. H. Jeans, Princeton University; Mr. A. R. Maxson, Columbia University; Professor J. F. Travis, Georgia School of Technology; Professor Vito Volterra, University of Rome; Miss Mary E. G. Waddell, Orono, Canada.

ERRATA.

On page 230, line 6, for $\frac{1}{4}\pi^2$ read $\frac{1}{8}\pi^2$.

On page 230, line 16, for $\frac{2\pi^2}{6} - \frac{\pi^2}{2.6} = \frac{\pi^2}{4}$, read $2.\frac{\pi^2}{12} - \frac{\pi^2}{2.12} = \frac{\pi^2}{8}$.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. FEBRUARY, 1906. No. 2.

THE GROUPS WHICH CONTAIN LESS THAN TWENTY OPERATORS OF ORDER THREE.

By G. A. MILLER.

All the operators of a given order which are contained in a group (G) generate an invariant subgroup if they do not generate the entire group. As the existence of invariant subgroups in G is generally very useful in determining other properties, it is of interest to know what groups contain a given number of operators of a certain order and are generated by these operators. Very little has been done along this line. One of the most interesting problems in this connection is the determination of all the groups which contain a small number of operators of order 2 and are generated by these operators. This problem has been solved in case the number of the operators of order 2 does not exceed 15.

The present note is devoted to a study of a few of the groups which seem almost as interesting as those just mentioned. It will be observed that very few theorems of group theory are employed in these considerations. According to a well known theorem due to Frobenius, the number of subgroups of order p in any group is of the form $1+kp$. Hence the groups which come under the present heading contain 2, 8, or 14 operators of order 3. In the first case the operators of order 3 generate an invariant subgroup of this order. The other two cases lead to much more interesting results.

§ 1. *Groups which contain just eight operators of order three and are generated by these operators.*

Let s_1, s_2, s_3, s_4 represent the generators of the four subgroups of order 3 which are contained in G. If one of these four operators is transformed into itself by some other one, the eight operators are contained in a group of order 9. If this is not the case each of them transforms the other three and hence it transforms among themselves the subgroups which they generate. Hence G transforms these four subgroups according to a transitive substitution group (T) of degree 4. The subgroup (H) of G which corresponds to the identity in T cannot contain any operator of order 3. From this it follows that the order of T is divisible by 3. That is, T is either the alternating or the symmetric group of degre 4.

It is evident that there is a (1, 1) correspondence between the operators of order 3 in G and the substitutions of order 3 in T. We may therefore suppose that the product $s_1 s_2$ corresponds to a substitution of order 2 in T. As $s_1 s_2$ is commutative with every operator of H it has either three or six conjugates under G. In the former case the three conjugates of $s_1 s_2$ with respect to its factors are identical. That is, $s_1 s_2$, $s_2 s_1$, $s_1^2 s_2 s_1^2$ are the same as $s_1 s_2$, $s_2^2 s_1 s_2^2$, $s_2 s_1$. Hence $s_1^2 s_2 s_1^2 = s_2^2 s_1 s_2^2$. The former is the inverse of $s_1 s_2 . s_2 s_1$* and the latter is the inverse of $s_2 s_1 . s_1 s_2$. Since the inverses of these products are equal the products are equal and the three conjugates $s_1 s_2$, $s_2 s_1$, $s_1^2 s_2 s_1^2$ are commutative; for, if two of a complete set of three conjugates are commutative the three must be commutative.

We have now proved that the conjugates of $s_1 s_2$ under G are commutative whenever $s_1 s_2$ has only three such conjugates. It remains to prove that $s_1 s_2$ is of order 2 in this case. From the equations $(s_1 s_2 . s_2 s_1)^2 = (s_1 s_2)^4$ and $(s_1 s_2 . s_2 s_1)^2 = (s_1^2 s_2 s_1^2)^{-2} = (s_1 s_2)^{-2}$ it follows that $s_1 s_2$ is of order 6. The equation $(s_1^2 s_2 s_1^2)^{-2}$ follows from the fact that if an operator transforms the α power of another operator into itself the α power of the second operator is equal to the α power of the transform. As $(s_1 s_2)^6 = 1$ and the order of H is not divisible by 3 it results that $s_1 s_2$ is of order two.

Since $s_1 s_2$, $s_2 s_1$ are commutative and of order 2 they generate the four-group. As their product is its own inverse, it is $s_1^2 s_2 s_1^2$. In other words, the four group generated by $s_1 s_2$, $s_2 s_1$ is transformed into itself by s_1, and s_1 has four conjugates under this four-group. These results may be stated as follows:

If a group contains only four subgroups of order 3 and if the product of two of its non-commutative operators of order 3 has only three conjugates under the group then these four subgroups generate the alternating group of order 12.

It remains to consider the case when $s_1 s_2$ has six conjugates under G. Two of the six conjugates correspond to the same substitution of T. These two are commutative and each of them transforms the remaining four, otherwise one of these six conjugates would be transformed into itself by more than one-sixth of the operators of G. As each of these six conjugates is commutative with just half of the operators of G which correspond to the four-group in T, it follows

*This is a special case of the theorem, If the product of two operators of order n is transformed successively by one of these operators, the continued product of the n conjugates thus obtained, in order, is the identity.

that the group generated by them has a commutator subgroup of order 2, which we shall represent by 1, c.

From the fact that $s_1s_2.s_2s_1$ is of the same order as $s_1^2s_2s_1^2$ and hence also of the same order as s_1s_2 it follows that the order of s_1s_2 is divisible by 4 but not by 8. In fact, $(s_1s_2.s_2s_1)^2=c(s_1s_2)^4$ is of the same order as $(s_1s_2)^2$ only when the order of s_1s_2 is divisible by 4 but not by 8. As $c(s_1s_2)^{-4}=(s_1^2s_2s_1^2)^2=(s_1s_2)^2$ it results that $c=(s_1s_2)^6$; and hence s_1s_2 is of order 4, the order of H being non-divisible by 3. The two conjugates s_1s_2, s_2s_1 must therefore generate the quaternion group. As this includes the inverse of $s_1^2s_2s_1^2$, it is transformed into itself by s_1. Moreover, s_1 has four conjugates under this group. These results may be expressed as follows:

If a group contains only four subgroups of order 3 the operators of this order generate the non-cyclic group of order 9, the alternating group of order 12, or the group of order 24 which does not include a subgroup of order 12. Hence it contains one of these subgroups invariantly.*

§ 2. *Groups which contain just fourteen operators of order three and are generated by these operators.*

If a group contains just fourteen operators of order 3 its Sylow subgroups must be cyclic since 7 is not of the form $1+p+kp^2$.† Hence the seven subgroups of order 3 form a single set of conjugates and are transformed by G according to a transitive group T of degree seven which contains just 7 subgroups of order 3. As in the preceding section there is evidently a (1, 1) correspondence between the operators of order 3 in G and the substitution of this order in T.

Two operators of order 3 in G, s_1, s_2 may be so selected that their product s_1s_2 corresponds to a substitution of order 7 in T. The three conjugates s_1s_2, s_2s_1, $s_1^2s_2s_1^2$ are commutative since T contains only one subgroup of order 7 and s_1s_2 is commutative with every operator of the subgroup H corresponding to the identity in T. From the equations $(s_1s_2.s_2s_1)^7=(s_1s_2)^{14}=(s_1^2s_2s_1^2)^{-7}=(s_1s_2)^{-7}$ it follows that $(s_1s_2)^{21}=1$. As H does not include any operator of order 3, s_1s_2 is of order 7.

The group generated by s_1s_2, s_2s_1 is either of order 7 or of order 49, since these operators are commutative. As this group includes $s_1^2s_2s_1^2$ it is transformed into itself by s_1. If its order is 7 the seven subgroups in question generate the group of order 21 which is defined by its order and the fact that it contains seven subgroups of order 3. As this group is so well known it remains only to prove that s_1s_2, s_2s_1 cannot generate a group of order 48. If this were the case s_1 would transform at least two of its 8 subgroups of order 7 into themselves. As it could not be commutative with the generators of both of these subgroups, it and one of these subgroups would again generate the given group of order 21. Hence it results that, *If a group contains just 14 operators of order 3 these operators generate the semi-metacyclic group of degree 7.*

* *Quarterly Journal of Mathematics*, Vol. 28 (1896), p. 274.

† *Transactions of The American Mathematical Society*, Vol. 6 (1905), p. 58.

INTERPRETATIONS OF THE IDENTICAL RELATIONS BETWEEN THE DETERMINANTS OF AN ARRAY.*

By R. P. BAKER. University of Iowa.

§1. $n=2$, $m=4$

(1) If the elements represent homogeneous coördinates of four points on a line, the 2×2 minors are proportional to the distances between pairs of points, and the identical relation in them gives Ptolemy's theorem.

$$(12)(34)-(13)(24)+(14)(23)=0.$$

The relations lineo-linear in elements and determinants such as,

$$a_1(a_2b_3)+a_2(a_3b_1)+a_3(a_1b_2)=0,$$
$$b_1(a_2b_3)+b_2(a_3b_1)+b_3(a_1b_2)=0,$$

give by linear combination Ptolemy's theorem for the points 1, 2, 3 and any other point in the line, or if this point be at infinity the identical relation between the distances of the three points 1, 2, 3.

(2) The elements represent homogeneous coördinates of two points A, B, in space. The determinants are the line coördinates of AB, and the identical relation the "fundamental relation" between them. The relations

$$z_1(a_2b_3)+z_2(a_3b_1)+z_3(a_1b_2)=0 \quad (z=a,\ b)$$

may be taken as a relation between the distances of any point in the line and the moments of the line with respect to the three planes which pass through a vertex.

If $z_1z_2z_3$ are current coördinates the relation is the equation of a plane through a vertex and the line.

§2. $n=2$, $m=5$.

The general formula gives three independent relations. For any eight elements and their six determinants we have the identity given in the case $n=2$, $m=4$. Thus we have the five relations,

$$A_1;\ (23)(45)-(24)(35)+(25)(34)=0,$$
$$A_2;\ (34)(51)-(35)(41)+(31)(45)=0,$$
$$A_3;\ (45)(12)-(41)(52)+(42)(51)=0,$$
$$A_4;\ (51)(23)-(52)(13)+(53)(12)=0,$$
$$A_5;\ (12)(34)-(13)(24)+(14)(23)=0.$$

*Sequel to a paper with similar title in the January number of the MONTHLY.

These relations are dependent in any case on some three of them, but the same three cannot always be taken as independent. There exist, in fact, five identities in the determinants:

$$A'_1;\ (12)A_2+(13)A_3-(41)A_4-(51)A_5=0,$$
$$A'_2;\ (23)A_3+(24)A_4-(52)A_5-(12)A_1=0,$$
$$A'_3;\ (34)A_4+(35)A_5-(13)A_1-(23)A_2=0,$$
$$A'_4;\ (45)A_5+(41)A_1-(24)A_2-(34)A_3=0,$$
$$A'_5;\ (51)A_1+(52)A_2-(35)A_3-(45)A_4=0.$$

The trivial case of all vanishing determinants being excluded, suppose that (12) is not zero. Then if A_3, A_4, A_5, vanish the relations A'_1, A'_2, show that A_1, A_2, also vanish, but the vanishing of any other three of the A's will not necessarily entail the vanishing of the remaining pair. There are also twenty lineo-linear relations.

The geometrical interpretation is similar to that in the case of a 2×4 matrix.

Trilinear Coordinates.

If the fundamental triangle be ABC and a_1 is the perpendicular distance from the point 1 to the side BC, to interpret $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ drop a perpendicular from C to the line 12 and let 12 meet BC at the angle α, AC at the angle β.

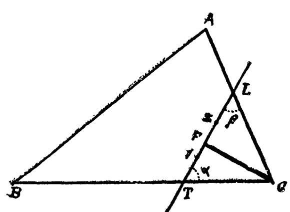

Fig. 4

Then $1T=a_1/\sin\alpha,\quad 2T=a_2/\sin\alpha,$
$1L=b_1/\sin\beta,\quad 2L=b_2/\sin\beta,$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\sin\alpha\ \sin\beta\begin{vmatrix} 1T & 1L \\ 2T & 2L \end{vmatrix}=\sin\alpha\ \sin\beta\begin{vmatrix} 1T & TL \\ 2T & TL \end{vmatrix}$$
$$=TL\ \sin\alpha\ \sin\beta\ \overline{(12)}.$$

Now $TL=TF-FL=CF(\cot\alpha+\cot\beta)$.

$\therefore\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=CF\sin C\ \overline{(12)}=$ (moment of line 12 with respect to C).$\sin C$.

To interpret $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we have the identities,

$$BC.a_1+CA.b_1+AB.c_1=2\triangle,$$
$$BC.a_2+CA.b_2+AB.c_2=2\triangle,$$
$$BC.a_3+CA.b_3+AB.c_3=2\triangle,$$

whence $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\dfrac{2.\triangle}{BC}\begin{vmatrix} b_1 & c_1 & 1 \\ b_2 & c_2 & 1 \\ b_3 & c_3 & 1 \end{vmatrix}=\dfrac{2.\triangle}{BC}[(b_1c_2)+(b_2c_3)+(b_3c_1)]$

$$=\frac{2.\triangle}{BC}[\overline{\overline{12}}.\sin A+\overline{\overline{23}}.\sin A+\overline{\overline{31}}.\sin A],$$

where $\overline{\overline{12}}$ denotes the moment of the line with respect to A. That is

$$(a_1\ b_2\ c_3)=\frac{2.\triangle\ \sin A}{BC}\ \text{Area } 123=\frac{4.\triangle^2}{AB.BC.CA}\ \text{Area } 123.$$

§3. $n=3$, $m=4$.

There are no relations between the determinants; there are three relations lineo-linear in the elements and determinants and twelve relations lineo-linear in the determinants and their 2×2 minors. To interpret in R_2,

(1) The lineo-linear relations between elements and determinants

$$x_1(a_2b_3c_4)-x_2(a_3b_4c_1)+x_3(a_4b_1c_2)-x_4(a_1b_2c_3)=0\quad [x=(a,\ b,\ c)].$$

The determinants are proportional to the areas of the triangles, the elements to the distances of the points from the sides of the reference triangle.

The relation being homogeneous in rows and columns proportionality factors need not be considered.

Making the usual convention as to signs of triangular areas, we have the theorem:

"In a complete quadrangle the sum of the products of the areas of the triangles into the distances of the remaining vertices from any line is zero."

If the line is at infinity this reduces to the areal identity 234. $-$341. $+$412. $-$123. $=0$.

(2) The relation,

$$(a_1b_2)(a_3b_4c_1)-(a_1b_3)(a_2b_4c_1)+(a_1b_4)(a_2b_3c_1)=0.$$

Proportionality factors being neglected as before, (a_1b_2) is the moment of the line 12 with respect to C multiplied by $\sin C$. Also $(a_1b_2c_3)$ is proportional to the area of the triangle 123.

The relation gives,

Moment $12\times$Area(341)$-$Moment $13\times$Area(241)$+$Moment $14\times$Area(231)$=0$,

which may be interpreted as follows:

"If three concurrent forces 12, 13, 14, be proportional to the length of the lines 12, 13, 14, and to the areas of the triangles 341, 421, 231 jointly and have their signs determined by the directions in which the triangles are described they are in equilibrium."

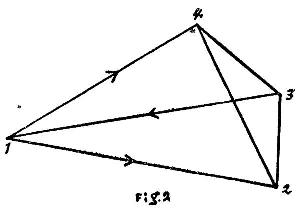

Fig. 2

In the case of 1234 being a parallelogram this reduces to the parallelogram of forces.

Or, expressing the areas of the triangles by the product of adjacent sides and sine of included angle, and the moments by the product of lengths of line and perpendicular, and expressing the perpendicular by the product of the line $C1$ and the sine of the proper angle we have, if angle $C12=P$, $213=Q$, $314=R$, and the line factors, now all common, be divided out,

$$\sin(P+Q+R)\sin Q-\sin(P+Q)\sin(Q+R)+\sin P\sin R=0,$$

a trigonometrical identity which reduces to the various forms of the addition theorem when $Q=\frac{1}{2}\pi$, $P=\frac{1}{2}\pi$.

NOTE ON FINDING THE COMPLEMENTARY FUNCTION OF A LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS WHEN THE AUXILIARY EQUATION HAS EQUAL ROOTS.

By B. F. FINKEL.

The following method, which is the one virtually implied though not expressly stated in Forsyth's *Differential Equations*, Art. 44, p. 64, has the advantage of simplicity and elegance of presentation, and is worthy, therefore, of being better known. Dr. G. E. Fisher, of the University of Pennsylvania, has used the method for some years in his classes. Dr. Fisher does not claim that the method is original with him, neither does he remember from whence the suggestion came to him.

As I am not aware that it has been published elsewhere, at least in none of the text books on the subject that I have examined, it is thought that its appearance in the MONTHLY will be helpful to many of its readers.

Let $y=Ae^{ax}+Be^{\beta x}+\dots\dots\dots+Le^{lx}$ be the *Complementary Function* of the general linear differential equation

$$\frac{d^ny}{dx^n}+A_1\frac{d^{n-1}y}{dx^{n-1}}+\dots\dots\dots+A_{n-1}\frac{dy}{dx}+A_ny=V.$$

Now if two of the roots are equal, as, for example, $a=\beta$, we have

$$y=(A+B)e^{ax}+Ce^{\gamma x}+\dots\dots\dots+Le^{lx}=A_1e^{ax}+Ce^{\gamma x}+\dots\dots\dots+Le^{lx}.*$$

We thus lack an arbitrary constant, there now being only $n-1$. In order to obtain the complete primitive, let us suppose, for the moment, that a and β are different, and that $\beta-a=h$. A particular solution would then be

*Forsyth's *Differential Equations*, 3rd edition, p. 64.

$$y = Ae^{\beta x} + Be^{\alpha x} = Ae^{(\alpha+h)x} + Be^{\alpha x},$$

for all values of A and B.

If $A = -B = \frac{1}{h}$, $y = \frac{1}{h}(e^{(\alpha+h)x} - e^{\alpha x})$ would still be a particular integral. When $h \doteq 0$, $\beta \doteq \alpha$, and $y = \frac{1}{h}(e^{(x+h)x} - e^{\alpha x})$ becomes a particular solution for the case of equal roots, that is,

$$y = \underset{h \doteq 0}{\mathrm{Lim}}\left[\frac{e^{(\alpha+h)x} - e^{\alpha x}}{h}\right] = \frac{d}{d\alpha}[e^{\alpha x}] = xe^{\alpha x}$$

is a particular solution when $\alpha = \beta$.

Hence, when two roots are equal, in order to find a second particular integral, we need only to obtain the derivative with respect to one of the equal roots, of the first particular integral.

Thus, if $\alpha = \beta$ and if $y = e^{\alpha x}$ is a particular integral, a second particular integral is $y = \frac{d}{d\alpha}[e^{\alpha x}] = xe^{\alpha x}$.

In the same way, we find a third particular integral, if three roots are equal. Thus, suppose $\alpha = \beta = \gamma$, and that $e^{\alpha x}$, $xe^{\alpha x}$ are two particular integrals. If we assume, for the moment, that β and γ are different and that $\gamma - \beta = k$, we would have as a particular integral $y = Cxe^{\gamma x} + Dxe^{\beta x}$, for all values of C and D.

If $C = -D = \frac{1}{k}$, $y = \frac{1}{k}(xe^{(\beta+k)x} - xe^{\beta x})$ would still be a particular integral. When $k \doteq 0$, $\gamma \doteq \beta$, and $y = \frac{1}{k}(xe^{(\beta+k)x} - xe^{\beta x})$ becomes a particular integral for the case when $\beta = \gamma$. But

$$y = \underset{k \doteq 0}{\mathrm{Lim}}\left[\frac{xe^{(\beta+k)x} - xe^{\beta x}}{h}\right] = \frac{d}{d\beta}[xe^{\beta x}] = x^2 e^{\beta x}.$$

Hence, if $\alpha = \beta = \gamma$, three particular integrals are $e^{\alpha x}$, $xe^{\alpha x}$, and $x^2 e^{\alpha x}$.

In the same way we find r particular integrals when r roots are equal. In the case of r equal roots, the *complementary function* is

$$y = A_1 e^{\alpha x} + A_2 \frac{d}{dx}(e^{\alpha x}) + A_3 \frac{d^2}{dx^2}(e^{\alpha x}) + A_4 \frac{d^3}{dx^3}(e^{\alpha x}) + \dots\dots\dots$$

$$+ A_r \frac{d^{r-1}}{dx^{r-1}}(e^{\alpha x}) + \dots\dots\dots + Le^{lx}.$$

Forsyth uses this method in solving example 6, p. 68, though he does not expressly state the method.

MULTIPLY PERFECT ODD NUMBERS WITH THREE PRIME FACTORS.

By R. D. CARMICHAEL.

The object of this note is to prove the following

PROPOSITION. There exist no multiply perfect odd numbers containing only three primes.

Let the numbers here considered be of the form $p_1^{a_1} p_2^{a_2} p_3^{a_3}$, where p_1, p_2, p_3 are distinct odd primes, and $p_1 < p_2 < p_3$. And also let the multiplicity be m, where $m > 1$. Now, by definition, the multiplicity times the number equals the sum of the factors. Hence,

$$(1) \qquad mp_1^{a_1} p_2^{a_2} p_3^{a_3} = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \frac{p_2^{a_2+1}-1}{p_2-1} \cdot \frac{p_3^{a_3+1}-1}{p_3-1}.$$

$$(2) \qquad m = \frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)} \cdot \frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)} \cdot \frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}.$$

$$(3) \qquad m < \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1} \cdot \frac{p_3}{p_3-1}.$$

The right member of (3) is greatest when the primes are smallest. By substituting 3, 5, 7 for p_1, p_2, p_3, we find that $m \overline{\gtrless} 2$. Now, with $m=2$ in (3), it is easily shown that we must have $p_1=3$, $p_2=5$, $p_3<16$, whence $p_3=7$, 11, or 13. Then (1) becomes

$$(4) \qquad 2^4 . 3^{a_1} . 5^{a_2} . p_3^{a_3}(p_3-1) = (3^{a_1+1}-1)(5^{a_2+1}-1)(p_3^{a_3+1}-1).$$

When $p_3=7$, equation (4) becomes

$$(5) \qquad 2^5 . 3^{a_1+1} . 5^{a_2} . 7^{a_3} = (3^{a_1+1}-1)(5^{a_2+1}-1)(7^{a_3+1}-1).$$

If a_3+1 is even, $7^{a_3+1}-1$ is divisible by 7^2-1, which contains 2^4. But, in any event, $5^{a_2+1}-1$ contains 2^2 and $3^{a_2+1}-1$ contains 2. The right member will then contain 2^7, which is impossible. Hence, a_3+1 is odd. Since odd powers of 7 end in 7 or 3, $7^{a_3+1}-1$ is not divisible by 5. This requires a_1+1 to be even; as odd powers of 3 end in 3 or 7, and $3^{a_1+1}-1$ is therefore not then divisible by 5. Since a_1+1 is even, $3^{a_1+1}-1$ is divisible by $3^2-1=2^3$. The right member of (5) then contains 2^6. This is impossible. Hence, there are no numbers of the type here considered when $p_3=7$.

Next, for $p_3=11$, (4) becomes

$$(6) \qquad 2^5 . 3^{a_1} . 5^{a_2+1} . 11^{a_3} = (3^{a_1+1}-1)(5^{a_2+1}-1)(11^{a_3+1}-1).$$

We can here, as above, show that a_3+1 is odd. Likewise, that a_1+1 is odd.

Then $3^{a_1+1}-1$ does not contain 5. Hence, the right member contains the factor 5 only in $11^{a_3+1}-1$. But this factor must occur at least twice, as $a_2 \neq 0$. By writing $(10+1)^{a_3+1}-1$ and expanding, we may easily show that it contains 5 only once unless a_3+1 is divisible by 5. Then, let $a_3+1=5n$. Now, $11^{5n}-1$ is divisible by 11^5-1, which contains a prime greater than 11. Hence, $p_3=11$ yields no numbers of the type here considered.

Finally, for $p_3=13$, (4) becomes

$$(7) \qquad 2^6.3^{a_1+1}.5^{a_2}.13^{a_3}=(3^{a_1+1}-1)(5^{a_2+1}-1)(13^{a_3+1}-1).$$

If a_3+1 is even, $13^{a_3+1}-1$ is divisible by 13^2-1. This introduces the inadmissible factor 7. Hence, a_3+1 is odd. The odd powers of 13 end in 3 or 7. Hence, $13^{a_3+1}-1$ is not now divisible by 5. If a_1+1 is odd, $3^{a_1+1}-1$ is not divisible by 5. But to satisfy the equation, it must contain 5. Hence, a_1+1 is even, and $3^{a_1+1}-1$ then contains the factor $3^2-1=2^3$. $5^{a_2+1}-1$ always contains 2^2, and $13^{a_3+1}-1$ always contains 2^2. Hence, the right member contains 2^7, which is impossible. Therefore, this case yields no numbers of the type here considered.

DEPARTMENTS.

ALGEBRA.

250. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Factor $a^2b^2(x^2+y^2)(a^2y^2+b^2x^2-a^2b^2)=(a^4y^2+b^4x^2)[\sqrt{(a^2y^2+b^2x^2)}+ab]^2$.

Solution by the PROPOSER.

Let $x=r\cos\theta$......... (1), $y=r\sin\theta$.........(2); then the given expression equated to zero becomes

$$(b^2\cos^2\theta+a^2\sin^2\theta)[a^2b^2-(b^4\cos^2\theta+a^4\cos^2\theta)]r^2$$
$$-2ab\sqrt{[(b^2\cos^2\theta+a^2\sin^2\theta)]}(b^4\cos^2\theta+a^4\sin^2\theta)r$$
$$=a^2b^2(b^4\cos^2\theta+a^4\sin^2\theta+a^2b^2)\text{.........}(3).$$

Multiplying both sides of (3) by the coefficient of r^2 and noticing that

$$a^2b^2-(b^4\cos^2\theta+a^4\sin^2\theta)=a^2b^2(\sin^2\theta+\cos^2\theta)-(b^4\cos^2\theta+a^4\sin^2\theta)$$
$$=b^2(a^2-b^2)\cos^2\theta+a^2(a^2-b^2)\sin^2\theta=(a^2-b^2)(b^2\cos^2\theta+a^2\sin^2\theta)\text{.........}(4),$$

and similarly,

$$a^2b^2+b^4\cos^2\theta+a^4\sin^2\theta=(a^2+b^2)(b^2\cos^2\theta+a^2\sin^2\theta)\text{.........}(5).$$

Completing the square, using the positive sign of the radical,

$$r(a^2-b^2)(b^2\cos^2\theta-a^2\sin^2\theta)=ab(a^2+b^2)\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}\ \ldots\ldots\ldots(6).$$

Multiplying both sides of (6) by $r(b^2\cos^2\theta-a^2\sin^2\theta)$, squaring, and putting in the values from (1) and (2),

$$(a^2-b^2)^2(b^2x^2-a^2y^2)^2-a^2b^2(a^2+b^2)^2(b^2x^2+a^2y^2)=0\ \ldots\ldots\ldots(7).$$

This is one factor of the given expression. Using the negative sign after completing the square in (3), and employing (4) and (5),

$$(a^2-b^2)\ (b^2\cos^2\theta-a^2\sin^2\theta)r\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}\ [r\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}+ab]$$
$$=0\ \ldots\ldots\ (8).$$

Equating the last factor to zero, rationalizing, using (1) and (2), we have $a^2y^2+b^2x^2-a^2b^2$ as a second factor.

Also solved by G. B. M. Zerr.

251. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}+\dfrac{1}{3(n+3)}+$ etc.,$=$

$$\frac{1}{n^2}+\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\ldots+\frac{1}{l(n-l)}\right],$$

l being equal to $n-1$, n being any positive integer greater than one.

Solution by L. E. NEWCOMB, Los Gatos, Cal.

The general term is, $\dfrac{1}{r(n+r)}=\dfrac{1}{nr}-\dfrac{1}{nr(r+n)}$. Let $r=1, 2, 3, \ldots\ldots\ldots$ in succession; then $\dfrac{1}{n+1}=\dfrac{1}{n}-\dfrac{1}{n(n+1)}$, $\dfrac{1}{2(n+2)}=\dfrac{1}{2n}-\dfrac{1}{n(n+2)}$, $\dfrac{1}{3(n+3)}$ $=\dfrac{1}{3n}-\dfrac{1}{n(n+3)}$.

$\therefore$ Sum $=\dfrac{1}{n}-\dfrac{1}{n(n+1)}+\dfrac{1}{2n}-\dfrac{1}{n(n+2)}+\dfrac{1}{3n}-\dfrac{1}{n(n+3)}\ldots\ldots\ldots$ and all the terms after the rth vanish.

$$\therefore \text{Sum}=\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}+\ldots+\frac{1}{n^2}\equiv\frac{1}{n^2}+\left[\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}+\ldots+\frac{1}{ln}\right]\ (1).$$

In the series (2) $\dfrac{1}{n-1}+\dfrac{1}{2(n-2)}+\dfrac{1}{3(n-3)}+\ldots+\dfrac{1}{l(n-l)}$, the general

term is $\frac{1}{r(n-r)}=\frac{1}{nr}+\frac{1}{n(n-r)}$, and since $\frac{1}{n-1}=\frac{1}{n}+\frac{1}{n(n-1)}$, $\frac{1}{2(n-2)}=\frac{1}{2n}+\frac{1}{n(n-2)}$, $\frac{1}{l(n-l)}=\frac{1}{ln}+\frac{1}{n(n-l)}\equiv\frac{1}{n(n-1)}+\frac{1}{n}$, the sum of (2)$=$ $\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}$..............$\frac{1}{n}+\left[\frac{1}{n(n-1)}+\frac{1}{n(n-2)}+.....+\frac{1}{n}\right]$. The terms within and without the parenthesis are now plainly identical; consequently, $\frac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+.....+\frac{1}{l(n-l)}\right]$ substituted for $\frac{1}{n}+\frac{1}{2n}+\frac{1}{3n}$ .... $\frac{1}{ln}$ in the right hand member of (1) will satisfy the equation.

II. Solution by R. D. CARMICHAEL, Hartselle, Alabama.

Represent the series of the first member by S_n. Then,

$$S_2=\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}.....=\tfrac{1}{2}[(1-\tfrac{1}{3})+(\tfrac{1}{2}-\tfrac{1}{4})+(\tfrac{1}{3}-\tfrac{1}{5})+(\tfrac{1}{4}-\tfrac{1}{6}).....]=\tfrac{1}{2}(1+\tfrac{1}{2}).$$

$$S_3=\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+.....=\tfrac{1}{3}[(1-\tfrac{1}{4})+(\tfrac{1}{2}-\tfrac{1}{5})+(\tfrac{1}{3}-\tfrac{1}{6})+(\tfrac{1}{4}-\tfrac{1}{7}).....]=\tfrac{1}{3}(1+\tfrac{1}{2}+\tfrac{1}{3}).$$

....

$$S_n=\frac{1}{n}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}.....+\frac{1}{n})=\frac{1}{n^2}+\frac{1}{n}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}.....+\frac{1}{n-1})$$

$$=\frac{1}{n^2}+\frac{1}{n}\left(\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}.....+\tfrac{1}{3}+\tfrac{1}{2}+1\right)$$

$$\therefore 2S_n=\frac{2}{n^2}+\frac{1}{n}\left[(1+\frac{1}{n-1})+(\tfrac{1}{2}+\frac{1}{n-2})+(\tfrac{1}{3}+\frac{1}{n-3}).....+(1+\frac{1}{n-1})\right].$$

$$\therefore S_n=\frac{1}{n^2}+\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+..........\frac{1}{l(n-l)}\right],\ l \text{ being equal to } n-1.$$

Also solved by G. W. Greenwood, Henry Heaton, and G. B. M. Zerr.

252. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x-y=\frac{1}{3}\pi$; (2) $\sin x=\cos^3 y$.

Solution by J. SCHEFFER, Hagerstown, Md.

Since $x=y+\frac{1}{3}\pi$, we have $\sin x=\frac{1}{2}\sin y+\frac{1}{2}\sqrt{3}\cos y$.

$\therefore \frac{1}{2}\sin y+\frac{1}{2}\sqrt{3}\cos y=\cos^3 y$.

$\therefore 1-\cos^2 y=4\cos^6 y-4\sqrt{3}\cos^4 y+3\cos^2 y$, or $4\cos^6 y-4\sqrt{3}\cos^4 y+4\cos^2 y-1=0$; putting $\cos^2 y=z$, we get $z^3-\sqrt{3}z^2+z-\frac{1}{4}=0$; putting $z=t+\frac{1}{3}\sqrt{3}$, we get $t^3=\frac{1}{4}-\frac{1}{9}\sqrt{3}$; $\therefore t=(\frac{1}{4}-\frac{1}{9}\sqrt{3})^{\frac{1}{3}}$.

$$\therefore y=\cos^{-1}\pm[\tfrac{1}{3}\sqrt{3}+(\tfrac{1}{4}-\tfrac{1}{9}\sqrt{3})^{\frac{1}{3}}]^{\frac{1}{2}},\ x=\sin^{-1}\pm[\tfrac{1}{3}\sqrt{3}+(\tfrac{1}{4}-\tfrac{1}{9}\sqrt{3})^{\frac{1}{3}}]^{\frac{3}{2}}.$$

The two acute angles x and y are very nearly* $71^\circ\ 1'$ and $11^\circ\ 1'$.

Also solved by R. D. Carmichael, S. A. Corey, Henry Heaton, G. B. M. Zerr.

AVERAGE AND PROBABILITY.

174. Proposed by HENRY HEATON, Atlantic, Iowa.

Chords are drawn through every point of the surface of a given circle in every possible direction. What is their average length?

Solution by J. EDWARD SANDERS, Reinersville, Ohio.

Let θ=the angle the chord makes with r, then its length is $2\sqrt{(a^2-r^2\sin^2\theta)}$.

$$\therefore \triangle=\frac{4}{\pi a^2}\int_0^{\frac{1}{2}\pi}d\theta\int_0^a 2\sqrt{(a^2-r^2\sin^2\theta)}rdr=\frac{8a}{3\pi}\int_0^{\frac{1}{2}\pi}\frac{1-\cos^3\theta}{\sin^2\theta}d\theta=\frac{16a}{3\pi}.$$

Also solved by G. B. M. Zerr, and the Proposer.

175. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a line l is divided into three parts by two points taken at random on it, what is the mean value of the triangle whose sides are equal to the three parts? (Only those cases are to be considered in which the the three parts will form a triangle.)

Remark by S. A. COREY, Hiteman, Iowa.

Assuming that by mean value is meant average area, the problem becomes identical with problem 156, Average and Probability, a solution of which appeared in the MONTHLY in December, 1904, on page 237. The average area as there given is† $\pi l^2/105$.

CALCULUS.

213. Proposed by EDWIN L. RICH, Schenectady, N. Y.

Let $f(x)$ be any function of x, and $f'(x)$ its derivative. If $u=[f'(x)]^{-\frac{1}{2}}$, $v=f(x)[f'(x)]^{-\frac{1}{2}}$, then $\dfrac{1}{u}\dfrac{d^2u}{dx^2}-\dfrac{1}{v}\dfrac{d^2v}{dx^2}=0$.

I. Solution by W. L. TRYON, Ithaca, N. Y.

If $u=[f'(x)]^{-\frac{1}{2}}$, $\dfrac{du}{dx}=-\tfrac{1}{2}[f'(x)]^{-\frac{3}{2}}f''(x)$,

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*By the use of graphical methods Dr. Westlund obtains $x=71^\circ\ 1'\ 27''$, $y=11^\circ\ 1'\ 27''$. Mr. Heaton's result, correct to the sixth decimal place, is $x=71^\circ\ 1'\ 28''$, $y=11^\circ\ 1'\ 28''$. G.

†Solutions were contributed by Henry Heaton, J. Edward Sanders, and G. B. M. Zerr. The problem is solved in Williamson's *Integral Calculus*, Seventh Edition, p. 359. G.

$$\frac{d^2u}{dx^2}=-\tfrac{1}{2}\{-\tfrac{3}{2}[f'(x)]^{-\frac{5}{2}}[f''(x)]^2+[f'(x)]^{-\frac{3}{2}}[f'''(x)]\},$$

$$\frac{1}{u}\frac{d^2u}{dx^2}=(-\tfrac{1}{2})\{-\tfrac{3}{2}[f'(x)]^{-2}[f''(x)]^2+[f'(x)]^{-1}[f'''(x)]\}.$$

Also, if $v=f(x)[f'(x)]^{-\frac{1}{2}}$, $\dfrac{dv}{dx}=[f'(x)]^{\frac{1}{2}}-\tfrac{1}{2}f(x)[f'(x)]^{-\frac{3}{2}}f''(x)$,

$$\frac{d^2v}{dx^2}=-\tfrac{1}{2}f(x)\{-\tfrac{3}{2}[f(x)]^{-\frac{5}{2}}[f''(x)]^2+[f'(x)]^{-\frac{3}{2}}[f'''(x)]\},$$

$$\frac{1}{v}\frac{d^2v}{dx^2}=-\tfrac{1}{2}\{-\tfrac{3}{2}[f(x)]^{-2}[f''(x)]^2+[f'(x)]^{-1}[f'''(x)]\}=\frac{1}{u}\frac{d^2u}{dx^2}.$$

$$\therefore\ \frac{1}{u}\frac{d^2u}{dx^2}-\frac{1}{v}\frac{d^2v}{dx^2}=0.$$

II. Solution by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Since $u=(y')^{-\frac{1}{2}}$, $v=y(y')^{-\frac{1}{2}}$, where $y'=f(x)$. $v=uy$.

$$\therefore\ v''=\frac{d^2v}{dx^2}=uy''+2u'y'+yu''.\quad \therefore\ \frac{v''}{v}=\frac{y''}{y}+\frac{2u'y'}{uy}+\frac{u''}{u}.$$

$$\therefore\ \frac{u''}{u}-\frac{v''}{v}=-\left[\frac{y''}{y}+\frac{2u'y'}{uy}\right]=0.$$

$\therefore uy''=-2u'y'$, but $u'=-\frac{1}{2}(y')^{-\frac{3}{2}}y''$. $\therefore -2u'y'=(y')^{-\frac{1}{2}}y''=uy''$. This proves the required result.

Also solved by F. Anderegg, R. D. Carmichael, S. A. Corey, J. Scheffer, and G. B. M. Zerr.

214. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Prove that $\pi^2=6.\dfrac{2^2}{2^2-1}.\dfrac{3^2}{3^2-1}.\dfrac{5^2}{5^2-1}$.......... where the squared numbers in the numerator are the natural *primes* in order.

Solution by WILLIAM HOOVER, Athens, Ohio.

We have from trigonometry, $\pi^2=6(1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+.....)$..........(1).

In a foot-note to page 109, Boole's *Finite Differences*, edition 1872, is the following:

$$\text{``}S\equiv 1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+.....;$$

$$\frac{1}{2^{2n}}S\equiv\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+.....;$$

$$\left(1-\frac{1}{2^{2n}}\right)S=1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\dots\dots;$$

$$\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)S=1+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\dots\dots;$$

....

all of the terms of the form $1/(2p)^{2n}$ being removed.

Ultimately, $\left[\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\dots\dots\dots\right]S=1.$"

Let $S=S_1$ when $n=1$; then equation (1) becomes

$$\pi^2=6S_1=6\,\frac{1}{1-\frac{1}{2^2}}\cdot\frac{1}{1-\frac{1}{3^2}}\cdot\frac{1}{1-\frac{1}{5^2}}\dots\dots=6\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\dots\dots\dots$$

It is plain into what general form all of Bernoulli's Numbers of the form given by Boole, viz:

$$B_{2n-1}=\frac{2(2n)!}{(2\pi)^{2n}}\left[1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\dots\right],$$

can be thrown.

Also solved by F. Anderegg, and G. B. M. Zerr.

GEOMETRY.

276. Proposed by G. I. HOPKINS, Manchester, N. H.

ABC is an equilateral triangle whose vertices are the centers of circles with radius AB, and H is the center of the arc AB. From F, the point of intersection of the circles whose centers are A and C, a line is drawn through H to the circumference CAN. Draw BN, and prove that the angle ABN is an angle of a regular pentagon.

Solution by G. B. M. ZERR. A. M., Ph. D.. Parsons, W. Va.

Let $BN=AB=AC=BC=a$. Then $BF=a\sqrt{3}$, $\angle ABF=\pi/6$, $\angle HFB=\pi/12$. $\therefore \angle HDB=\pi/4$.

$BN : BF=\sin\pi/6 : \sin N$; $a : a\sqrt{3}=\sin\pi/6 : \sin N$.

$\therefore \sin N=\sqrt{3}\sin\pi/6=\dfrac{3-\sqrt{3}}{2\sqrt{2}}=.44828.$

$\therefore N=26°\ 38'$. $\therefore ABN=108°\ 22'$, or $22'$ larger than the angle of a regular pentagon. The construction thus gives merely a rough approximation.

Also solved by L. E. Newcomb, and J. Scheffer.

277. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It is tacitly assumed in elementary geometry that as the number of sides of a regular polygon inscribed in a circle is increased, *in any manner*, that its perimeter has a fixed limit. Beginning with a square and then continually doubling the number of sides we get for the perimeter $2^{n+2}\sqrt{[2-E^n(0)]}$, where $E(x)\equiv\sqrt{(2+x)}$. Beginning with a hexagon we get $2^{m+1}3\sqrt{[2-E^m(1)]}$. The definition of the length of a circle assumes that these expressions have the same limit as $n\doteq\infty$ and $m\doteq\infty$. Prove it.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Putting $\cos x=b$, we have $\sin\frac{1}{2}x=\sqrt{[\frac{1}{2}(1-b)]}=\frac{1}{4}\sqrt{[2-2b]}$, $\cos\frac{1}{2}x=\frac{1}{2}\sqrt{[2+2b]}$, $\sin\frac{1}{4}x=\frac{1}{2}\sqrt{[2-\sqrt{(2+2b)}]}$, $\cos\frac{1}{4}x=\frac{1}{2}\sqrt{[2+\sqrt{(2+2b)}]}$, $\sin\frac{1}{8}x=\frac{1}{2}\sqrt{\{2-\sqrt{[2+\sqrt{(2+2b)}]}\}}$, etc. Let the radius of a circle$=1$, then we may represent the perimeter of a regular polygon by $2^{n+2}\sin\frac{1}{2^n}x$. For $n=\infty$, this assumes the indeterminate form $\infty\doteq0$; but we may reduce it to $\dfrac{2^2\sin(1/2^n)x}{1/2^n}$. Differentiating numerator and denominator with reference to n, we get $2^2x\cos(1/2^n)x$. For $n=\infty$, this becomes 2^2x. For $x=90°$, $2^2\times90=360°$.

In regard to the hexagon, we have similarly, $\dfrac{3\sin(1/2^n)\times120°}{1/2^n}$, which after differentiation as above, becomes $3\times120°=360°$, so that the limit in both cases is $360°$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$E^n(0)=\sqrt{\{2+\sqrt{[2+\sqrt{(2+\sqrt{(2+, \text{etc.}))}]}\}}$;

$E^m(1)=\sqrt{\{2+\sqrt{[2+\sqrt{(2+\sqrt{(2+, \text{etc.}))}]}\}}$, the last term in the root being $\sqrt{3}$.

Let $E^n(0)=E^m(1)=x$. $\therefore\sqrt{(2+x)}=x$, or $x=2$.

$$\frac{2^{n+2}\sqrt{[2-E^n(0)]}}{2^{m+1}3\sqrt{[2-E^m(1)]}}=\frac{2^{n+2}\sqrt{[2-x]}}{2^{m+1}\sqrt{[18-9x]}}=\frac{2^{n+1}\sqrt{[8-4x]}}{2^{m+1}\sqrt{18-9x]}}.$$

$$\frac{2^{n+1}}{2^{m+1}}\cdot\frac{\sqrt{[8-4x]}}{\sqrt{[18-9x]}}=\frac{\sqrt{[8-4x]}}{\sqrt{[18-9x]}},\text{ when } n=m=\infty.$$

If $\dfrac{\sqrt{[8-4x]}}{\sqrt{[18-9x]}}=1$, $8-4x=18-9x$, or $x=2$,

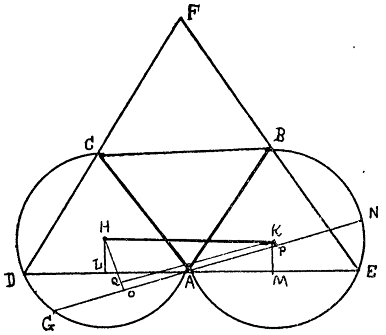
the same value as found above for x. Hence the expressions are equal when n and m are indefinitely increased.

279. Proposed by C. C. WENTWORTH, C. E., Roanoke, Va.

To construct geometrically the maximum equilateral triangle circumscribed about a given triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle. On AC, AB describe segments to contain angles of 60°. Let H, K be the centers of these circles. Join HK and parallel to HK draw DAE. Join DC and EB, and produce DC, EB until they meet in F. Then FDE is the required maximum equilateral triangle.



$$\angle D = \angle E = \angle F = 60^\circ.$$

Draw any other line GN through A. Let fall the perpendiculars HL, KM, HO, KP, and draw KQ parallel to GN.

$$KQ = OP = \tfrac{1}{2}GN;\ HK = LM = \tfrac{1}{2}DE,\ HK > KQ. \quad \therefore DE > GN.$$

$\therefore DE$ is the longest line that can be drawn through A.

$\therefore FDE$ is the maximum equilateral triangle required.

The same method will serve to describe the maximum triangle having any angles, about ABC.

Also solved by J. Scheffer.

MISCELLANEOUS.

155. Proposed by A. H. HOLMES, Brunswick, Maine.

There are two vessels, one containing a gallons of alcohol, the other b gallons of water. Suppose that c gallons are simultaneously taken from each and poured into the other, how many times must this be done so that there will be the same proportion of alcohol to water in each vessel?

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let $F(x)$ represent the number of gallons in the first vessel after x operations, then the number of gallons in the first vessel after the next operation will be $F(x)-\dfrac{c}{a}F(x)+\dfrac{c}{b}[a-F(x)]$. $\quad\therefore F(x+1)-[1-(\dfrac{c}{a}+\dfrac{c}{b})]F(x)=\dfrac{ac}{b}$.

By the calculus of finite differences, we get

$$F(x)=\frac{a^2}{a+b}+C[1-(\frac{c}{a}+\frac{c}{b})]^x.$$

Since for $x=0$, $F(x)=a$, we get $C=\dfrac{ab}{a+b}$.

$$\therefore F(x)=\frac{a^2}{a+b}+\frac{ab}{a+b}[1-(\frac{c}{a}+\frac{c}{b})]^x.$$

It is easily seen that when the same proportion of alcohol to water prevails, the contents of the alcohol in the first vessel will be$=a^2/(a+b)$.

$\therefore x$ must be$=\infty$.

II. **Solution by G. B. M. ZERR. A. M., Ph. D.. Parsons, W. Va.**

After the first operation, there are $a-c$ gallons of alcohol in the first vessel, and c gallons of alcohol in the second vessel. After the second operation, there are $a-2c+c^2(1/a+1/b)$ gallons of alcohol in the first vessel, and $2c-c^2(1/a+1/b)$ gallons of alcohol in the second vessel.

Let $A=c(1/a+1/b)$. Then, after the third operation, there are $a-3c+3Ac-A^2c$ gallons of alcohol in the first vessel, and $3c-3Ac+A^2c$ gallons of alcohol in the second vessel. After the nth operation there are

$$a-nc+\frac{n(n-1)}{2!}Ac-\frac{n(n-1)(n-2)}{3!}A^2c+\dots\pm A^{n-1}c=a+\frac{c(1-A)^n-c}{A}$$ gallons of

alcohol, and $\dfrac{c-c(1-A)^n}{A}$ gallons of water in the first vessel, and

$$nc-\frac{n(n-1)}{2!}Ac+\frac{n(n-1)(n-2)}{3!}A^2c-\dots\pm A^{n-1}c=\frac{c-c(1-A)^n}{A}$$ gallons of alcohol, and $b+\dfrac{c(1-A)^n-c}{A}$ gallons of water in the second vessel.

$$\therefore \frac{Aa+c(1-A)^n-c}{c-c(1-A)^n}=\frac{c-c(1-A)^n}{Ab+c(1-A)^n-c}.$$

$$\therefore (1-A)^n=\frac{c(a+b)-Aab}{c(a-b)}=0. \quad \therefore n=-\infty, \text{ or } A=1.$$

$\therefore$ The result stated can only happen when $a=b=2c$, then $n=1$.

156. **Proposed by R. D. CARMICHAEL, Hartselle, Ala.**

There exist no multiply perfect odd numbers of multiplicity n containing only n distinct primes.

Solution by JACOB WESTLUND, Ph. D., Purdue University, Lafayette, Ind,

If n denotes the multiplicity of a multiply perfect number $p_1^{a_1}p_2^{a_2}\dots p_i^{a_i}$, where $p_1, p_2, \dots$ are distinct primes, we have

$$n=\frac{p_1-\dfrac{1}{p_1^{a_1}}}{p_1-1}\cdot\frac{p_2-\dfrac{1}{p_2^{a_2}}}{p_2-1}\dots, \text{ and hence } n<\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\dots\frac{p_i}{p_i-1}.$$

Now, if $p_1>2$, we have

$$\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\ \cdots\cdots\ \frac{p_i}{p_i-1}\leqq\frac{3}{2}\cdot\frac{4}{3}\ \cdots\cdots\ \frac{i+2}{i}=\frac{i+2}{2}.$$

Hence we should have $n<\frac{i+2}{2}$, or $i>2n-1$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

256. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Three men, A, B, and C, rented a pasture for a fixed amount, each to pay per month in proportion to the stock pastured. During the first month A put in 3 horses and B and C each some horses, and B paid for the month $6, but A and C each defaulted payment. During the next month each put in one more horse, and C paid for the month $7.20, but A and B each defaulted payment. During the next month each put in one more horse, and A paid his bill for the month, $5, but B and C each defaulted.

Required: (1) the rent of the pasture per month; (2) the number of horses B and C each put in during the first month; and (3) the amount A, B, and C, each, owed for the unpaid service.

257. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x+y=10$, (2) $3x=\log_{10}y$.

258. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Sum the infinite series $\frac{n^2}{(4n^2-1)^2}$ beginning with $n=1$, n being always odd.

CALCULUS.

216. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Find the limit of the sum of the series

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+m^2},$$

when n and m are indefinitely increased. (Distinguish the several cases arising from the different *relative* values of m and n.)

217. Proposed by S. A. COREY, Hiteman, Iowa.

In *The Analyst*, Vol. II, p. 120, 1875, G. W. Hill finds by the method of mechanical quadrature the value of $\int_0^{\frac{1}{2}\pi}\frac{xdx}{\sin x[1+.16\cos^2 x]^{\frac{3}{2}}}$ to be 1.6576363.

Evaluate the definite integral by some other method and verify above result.

DIOPHANTINE ANALYSIS.

132. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

From the numbers, 0, 1, 2, ..., 42, select seven, such that the 42 differences of these seven numbers shall be congruent (mod. 43) to the numbers 0, 1, 2, ..., 42. The differences may be both $+$ and $-$.

133. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Find all perfect numbers of four primes and of multiplicity 4.

GROUP THEORY.

14. Proposed by O. E. GLENN, Springfield, Mo.

Hölder has proved* that any group (G) of order $\sum_{i=1}^{n} p_i$ (p_i a prime$\neq p_j$) may be generated as follows: $M^\mu=N^\nu=1$, $N^{-1}MN=M^a$, where $\{M\}$ is the product of all the invariant subgroups of G of prime order and $\{N\}$ is any one of a set of conjugate cyclical subgroups of order ν, ($\sum_{i=1}^{n} p_i=\mu\nu$). Find the generating relations of G in terms of operations of prime order, and express M and N in terms of these operations, for $n=4$.

MECHANICS.

187. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

188. Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in the Educational Times, London.)

Spherical bubbles are rising in water. Find the relation between radius and velocity.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Calculus, 209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

Calculus, 212. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Show that any root of the equation $y^5-5y=4x$ satisfies the differential equation $\dfrac{d^4y}{dx^4}=(\frac{4}{5})^4\, x^{-3}\dfrac{d^4(x^{-\frac{1}{5}}y)}{d(x^{-\frac{4}{5}})^4}$. Generalize the problem.

*See Burnside, *Theory of Groups*, p. 353.

NOTES AND NEWS.

At Syracuse University, Professor O. S. Stetson has been promoted to an assistant professorship of mathematics.

Popular Science Monthly of February contains an article by Professor G. A. Miller on "Some recent tendencies in mathematical instruction."

At Rutgers College, Mr. Richard Morris has been promoted from an instructorship to an associate professorship in graphics and applied mathematics.

Mr. Charles Haseman, assistant in mathematics at Indiana University, has obtained leave of absence and will spend the summer in graduate study at Göttingen.

Bowdoin College has received from Colonel I. H. Wing of Batfield, Wisconsin, a fund of $50,000, to endow the chair of mathematics. Colonel Wing is an alumnus of Bowdoin.

Professor C. A. VanVelzer, head professor of mathematics at the University of Wisconsin, has resigned, the resignation to take effect July 1, 1906. He intends to retire from teaching.

The San Francisco Section of The American Mathematical Society met at Stanford University on February 24. The printed program announced fifteen papers to be read at the meeting.

Mr. H. W. Reddick, a graduate of Indiana University, and until February first, fellow in the University of Illinois, has been appointed to a position in Shortridge High School at Indianapolis.

A bill granting permission to Professor Simon Newcomb, U. S. N., to accept the decoration of the order 'Pour le Mérite, für Wissenschaften und Kunste,' tendered by the emperor of Germany, passed the senate on February 8.

Professor H. Poincare's address, "The Present and Future of Mathematical Physics," delivered before the section of applied mathematics at the St. Louis International Congress of 1904, is published in the February *Bulletin of the American Mathematical Society*. The translation is by Professor J. W. Young.

We learn from the *Bulletin of the American Mathematical Society* that a memorial to the late Dr. George Salmon, Provost of Trinity College, Dublin, was unveiled Friday, January 5, in the National Cathedral of St. Patrick's, Dublin. It consists of two windows in one of the chapels, a portrait of Dr. Salmon, and a Latin inscription bearing testimony to his mathematical and theological work.

Dr. W. B. Smith, professor of mathematics, Tulane University, is an author of wide reputation in non-mathematical fields. His latest book, "The Color Line," is reviewed in the July (1905) *Monist* by Professor C. J. Keyser. Says the reviewer:

"The book in hand is the first of its kind to be written by a mathematician; and all the qualities of the mathematical mind, excepting that of proverbial dryness, are evident throughout, in its grasp and penetration, in the clearness and steadiness of its vision, in the sharp precision with which its problems are stated, and in the boldness, energy, and relentless logical rigor with which they are handled. * * * From beginning to end the appeal is from the individual standard to the race standard; from the traditional maxims however kindly, to the warning dictates of science however stern and cold; from the relative impotence of education to the 'omniprepotence of heredity.' * * * Whether one does or does not agree with Professor Smith's conclusions, the candid reader will allow the book is one with which future discussions of its difficult problems will be compelled to reckon."

WASHINGTON MATHEMATICS TEACHERS. The regular annual meeting of the Association of Teachers of Mathematics in Washington was held on December 29 at North Yakima. The following program was rendered at the meeting:

"Preliminary report on a proposed text of high school mathematics," by Professor Robert E. Moritz, University of Washington; "Present day defects in the teaching of secondary mathematics," by Mr. J. C. Keith, Seattle High School; "The relation of mathematics to applied science," by Professor O. L. Waller, Washington State College; "A comparison of some definitions with reference to the derivation of operations from them," by Professor W. A. Bratton, Whitman College; "Mathematics in the high school: Its subject matter and methods," by Mr. J. L. Dunn, Spokane High School

The following officers were elected for the ensuing year: President, Professor Robert E. Moritz, University of Washington; Vice President, Mr. J. C. Keith, Seattle High School; Secretary-Treasurer, Miss Zella E. Bisbee, North Yakima High School.

PORTRAITS OF EMINENT MATHEMATICIANS. Part II of the Portfolio of Portraits of Eminent Mathematicians, which is edited by Professor David Eugene Smith, has been issued by the Open Court Publishing Company of Chicago. The installment forms a beautiful collection of pictures, if possible surpassing in interest and value the notable collection forming Part I. The set includes twelve portraits on imperial Japanese vellum, 11 x 14, each portrait being accompanied by a brief biographical sketch, with occasional notes of interest concerning the artists represented. The collection includes: Pierre Simon Laplace, from a painting by Naigeon; Blaise Pascal, after an etching; Guillaume de l' Hopital, from a painting by Foucher; Niccolo Tartaglia, from an old engraving; Jakob Bernoulli, after an etching by Dupin; Gaspard Monge, from a lithograph by Delpech; Joseph Louis Lagrange, after an

engraving by Hart; Bonaventura Cavalieri, after a drawing by Alfieri; Johann Bernoulli, from a painting by Rüber; Leonhard Euler, from a painting by Darbes; Isaac Barrow, from the statue by Noble; Carl Frederich Gauss, from a painting by Jensen.

This series of portraits should adorn the walls of every room where mathematics is taught.

CURRENT PUBLICATIONS. The opening (January) number of the *Transactions of the American Mathematical Society* contains the following contributions: "On the relation between the three parameter groups of a cubic space curve and a quadric surface," by A. B. Coble; "On certain hyperabelian functions which are expressible by theta series," by J. I. Hutchinson; "On the form of a plane quintic curve with five cusps," by P. Field; "The symbolic treatment of differential geometry," by A. W. Smith; "Groups whose orders are powers of a prime," by W. B. Fite; "Differential parameters of the first order," by H. Maschke; "The Kronecker-Gaussian Curvature of hyperspace," by H. Maschke; "Groups containing only three operators which are squares," by G. A. Miller; "Theorems converse to Riemann's on differential equations," by D. R. Curtiss; "General mean value and remainder theorems," by G. D. Birkhoff; "Determination of the abstract groups of order p^2qr, p, q, r being distinct primes," by O. E. Glenn; "Note on the differential invariants of a surface and of space," by C. N. Haskins; "On improper multiple integrals," by James Pierpont.

The current number of the *American Journal of Mathematics* contains, "On the quaternary linear homogeneous groups modulo p of order a multiple of p," by L. E. Dickson; "On the integration of a system of differential equations in kinematics," by J. Eiesland; "On the determination of the properties of the nodal curve of a unicursal ruled surface," by C. H. Sisam; "Certain surfaces with plane or spherical lines of curvature," by L. P. Eisenhart; "The motion of a solid in an infinite liquid," by A. G. Greenhill.

The January number of the *Annals of Mathematics* contains the following papers: "The harmonic analysis of the semicircle and of the ellipse," by A. E. Kennelly; "Note on the possible number of operators of order 2 in a group of order 2^m," by G. A. Miller; "A geometrical problem connected with the continuation of a power series," by H. Maschke; "On the determination of a catenary with given directrix and passing through two given points," by H. F. MacNeish; "Concerning the discontinuous solution in the problem of the minimum surface of revolution," by H. F. MacNeish; "Introduction to the theory of Fourier series," by M. Bôcher.

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The Paris Academy of Sciences announces the Gutzman Prize of 100,000 francs for communication with any star or planet other than Mars. No affront to the Martians is intended.

Mr. John A. Bryan of Forney, Texas, has published a little booklet called *Perplex Problems.* It is a compilation of arithmetic, algebra, trigonometry, and miscellaneous problems, "many of which" the author says are "odds and ends and old puzzlers." The book has 38 pages, including six pages of answers, and the price is ten cents. The following are specimens:

"From a 50 gallon cask full of wine 10 gallons are drawn. The cask is then refilled with water. This is done ten times: how many gallons of wine remain?"

"A house and barn are 20 rods apart; the house is 10 and the barn 6 rods from a straight brook; what is the length of the shortest path by which one can go from the house to the brook and take water to the barn?"

The curious French and German poetical tributes to Archimedes, quoted in our November issue have called forth the following:

3 1 4 1 5 9
Now I, even I, would celebrate
2 6 5 3 5
In rhymes inapt, the great
8 9 7 9
Immortal Syracusan, rivaled nevermore,
3 2 3 8 4
Who in his wondrous lore,
6 2 6
Passed on before,
4 3 3 8 3 2 7 9
Left men his guidance how to circles mensurate.

[Adam C. Orr in the *Literary Digest.*]

3 1 4 1 5 9 2 6
Now, O hero, a great advancing in method,
5 3 5 8 9 7 9
Which you would proclaim wonderful, worketh universal;
3 2 3 8 4 6 2 6
Yet in our memories your labour is rooted;
4 3 3 8 4 2 7 9
Unto the end shouldst thou be amongst immortals!

[R. D. Carmichael].

ERRATUM.

Page 21, line 21, for $V_n = a' T^{n-1} \frac{\sin n\theta}{\sin\theta}$, read $U_n =$ etc.

$$\lambda\equiv\mu\equiv\nu\equiv0, \text{ or } \frac{p-1}{3} \text{ or } \frac{2(p-1)}{3} \pmod{p-1}.$$

If $p\equiv2 \pmod 3$, $n_3=1$, viz., $\lambda\equiv\mu\equiv\nu\equiv0 \pmod{p-1}$. Next when two are equal so that the congruence is $\lambda+2\mu\equiv0 \pmod{p-1}$, μ cannot be $\equiv0$, $\frac{p-1}{3}$, or $\frac{2(p-1)}{3}$ for then it would be $\equiv\lambda$. With these exceptions μ can have any value, and for each value of μ the congruence gives one value of λ. Hence if $p\equiv1 \pmod 3$, $n_2=p-4$, and when $p\equiv2 \pmod 3$, $n_2=p-2$.

If account be taken of the order of λ, μ, ν the total number of solutions of the congruence is $(p-1)^2$ since each unknown may assume $p-1$ values and when two are fixed the other is determined by the congruence. In terms of n_1, n_2 and n_3 the above totality of solutions equals $6n_1+3n_2+n_3$. Hence $6n_1+3n_2+n_3=(p-1)^2$, so that if $p\equiv1 \pmod 3$, $n_1=\frac{1}{6}(p^2-5p+10)$, and when $p\equiv2 \pmod 3$, $n_1=\frac{1}{6}(p^2-5p+6)$. Finally, when $p\equiv1 \pmod 3$, the total number (N) of solutions, disregarding order, of the given congruence is

$$N=3+p-4+\tfrac{1}{6}(p^2-5p+10)=\tfrac{1}{6}(p^2+p+4),$$

and if $p\equiv2 \pmod 3$,

$$N=1+p-2+\tfrac{1}{6}(p^2-5p+6)=\tfrac{1}{6}(p^2+p).$$

GEOMETRY.

278. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

AF, MN are parallel lines indefinitely extended toward FN; at right angles to AF, MN is AM of length 22; upon the base AB, which is in line with AM, is the triangle ABC whose sides are $AB=21$, $BC=10$, $AC=17$; find the sides of the maximum similar triangle with base extending from B to some point in AF, the vertex in line with MN.

Solution by G. B. M. ZERR. A. M., Ph. D.. Parsons, W. Va.

Let D be the vertex in AF, E the vertex in MN, $AD=x$, $ME=y$. Two cases are possible, BD the short side or BD the long side. The latter gives the maximum. Let $BD=21M$, $DE=10M$, $BE=17M$. Then

$$441M^2=441+x^2 \text{ ....} (1),$$

$$289M^2=1+y^2 \text{ ....} (2),$$

$$100M^2=(x-y)^2+484 \text{ ....} (3).$$

Substitution of x, y from (1) and (2) in (3) gives $\sqrt{[(M^2-1)(289M^2-1)]}=15M^2+1$. Hence $M^2=5$, $M=\sqrt{5}$, and the sides are $21\sqrt{5}$, $17\sqrt{5}$, $10\sqrt{5}$.

Also solved by the Proposer.

Let H, of order h, be any subgroup of G, and arrange the elements of the first column of $\Theta(x)$ with respect to the divisions $s_i H$. Then $\Theta(x)$ can be divided up into squares of h elements each, such that in any square the sums of the elements of the h rows are the same. For, the subscripts of the elements in the square containing the ith row and the jth column are contained in $s_i H.Hs_j^{-1}$, the subscripts for the different rows being obtained by taking the different operators in the first H, and those for the different columns by taking the different operators in the second H.

We now form the determinant Θ_1, of degree n/h, by taking for each element the sum of the elements in any row of the corresponding square of $\Theta(x)$. If we arrange the elements of the determinant of the symmetric group of degree 3 with respect to the subgroup $[1, (ab)]$, the corresponding Θ_1 is

$$\begin{vmatrix} x_{s_1}+x_{s_4} & x_{s_3}+x_{s_5} & x_{s_2}+x_{s_6} \\ x_{s_2}+x_{s_5} & x_{s_1}+x_{s_6} & x_{s_3}+x_{s_4} \\ x_{s_3}+x_{s_6} & x_{s_2}+x_{s_4} & x_{s_1}+x_{s_5} \end{vmatrix}.$$

For convenience I shall hereafter omit the x's and write only the subscripts.

I wish to prove that *Θ_1 is a factor of Θ*. (a) Add the elements of the first h rows of Θ to the corresponding elements of the first row; the elements of the second h rows to the corresponding elements of the $(h+1)$th row; and so on. (b) Now so re-arrange the rows and columns that the minor formed by the first h rows and columns is Θ_1. (c) Then multiply the elements of every row, except the first h rows, by h. (d) After this, subtract the elements of the first row from the corresponding elements of each row that was originally among the first h rows; the elements of the second row from the corresponding elements of each row that was originally among the second h rows; and so on. (e) Now to the elements of the first column add the corresponding elements of the columns that were originally the first h columns; to the elements of the second column add the corresponding elements of the columns that were originally among the second h columns; and so on. In the final form of the determinant each element of the first diagonal square of order h is h times the corresponding element of Θ_1 and all the other elements of the first h columns are zero. Moreover, a further obvious modification shows that the remaining elements of the first h rows can all be made zero. Therefore Θ_1 is a factor of Θ.

I shall illustrate the steps of this proof by the determinant of the symmetric group of degree 3.

$$\Theta = \begin{vmatrix} s_1 & s_4 & s_3 & s_5 & s_2 & s_6 \\ s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\ s_2 & s_5 & s_1 & s_6 & s_3 & s_4 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3 & s_6 & s_2 & s_4 & s_1 & s_5 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix}$$

$$
\text{(a)}\quad
\begin{vmatrix}
s_1+s_4 & s_4+s_1 & s_3+s_5 & s_5+s_3 & s_2+s_6 & s_6+s_2 \\
s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\
s_2+s_5 & s_5+s_2 & s_1+s_6 & s_6+s_1 & s_3+s_4 & s_4+s_3 \\
s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\
s_3+s_6 & s_6+s_3 & s_2+s_4 & s_4+s_2 & s_1+s_5 & s_5+s_1 \\
s_6 & s_3 & s_4 & s_2 & s_5 & s_1
\end{vmatrix}
$$

$$
\text{(b), (c)}\quad
\begin{vmatrix}
s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\
s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\
s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\
2s_4 & 2s_5 & 2s_6 & 2s_1 & 2s_3 & 2s_2 \\
2s_5 & 2s_6 & 2s_4 & 2s_2 & 2s_1 & 2s_3 \\
2s_6 & 2s_4 & 2s_5 & 2s_3 & 2s_2 & 2s_1
\end{vmatrix}
$$

$$
\text{(d)}\quad
\begin{vmatrix}
s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\
s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\
s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\
s_4-s_1 & s_5-s_3 & s_6-s_2 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\
s_5-s_2 & s_6-s_1 & s_4-s_3 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\
s_6-s_3 & s_4-s_2 & s_5-s_1 & s_3-s_6 & s_2-s_4 & s_1-s_5
\end{vmatrix}
$$

$$
\text{(e)}\quad
\begin{vmatrix}
2(s_1+s_4) & 2(s_3+s_5) & 2(s_2+s_6) & s_4+s_1 & s_5+s_3 & s_6+s_2 \\
2(s_2+s_5) & 2(s_1+s_6) & 2(s_3+s_4) & s_5+s_2 & s_6+s_1 & s_4+s_3 \\
2(s_3+s_6) & 2(s_2+s_4) & 2(s_1+s_5) & s_6+s_3 & s_4+s_2 & s_5+s_1 \\
0 & 0 & 0 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\
0 & 0 & 0 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\
0 & 0 & 0 & s_3-s_6 & s_2-s_4 & s_1-s_5
\end{vmatrix}
$$

If H' is conjugate to H in G, the corresponding factor Θ'_1 is the same as Θ_1. For, the element in the ith row and jth column of Θ_1 is Σs_a, where s_a runs through the operators of $s_i H s_j^{-1}$. If $s_r H s_r^{-1} = H'$, the element in the ith row and jth column of Θ_1 is the same as the element in the kth row and lth column of Θ'_1, where $s_i = s_k s_r$, $s_j^{-1} = s_r^{-1} s_l^{-1}$, since $s_i H s_j^{-1} = s_k s_r H s_r^{-1} s_l^{-1} = s_k H' s_l^{-1}$. Moreover, s_j fixes s_l, and therefore the elements of the jth column of Θ_1 are the same, except as to arrangement, as those of the lth column of Θ'_1. Likewise the elements of the ith row of Θ_1 are the same, except as to arrangement, as those of the kth row of Θ'_1. Therefore $\Theta'_1 = \pm\Theta_1$.

A GENERAL METHOD OF DEDUCING THE EQUATION OF A TANGENT TO A CURVE.

By PROFESSOR G. W. GREENWOOD, M. A., McKendree College.

The following method of finding the equation of a tangent to a curve is more general than those usually given in texts in elementary analytical geometry, but can readily be substituted for any of them. At the same time it has many additional advantages.

Through any point $P\equiv(x', y')$ draw a line l making an angle θ with OX. If this line has a point, or points, in common with a locus given by a rational integral equation, denote such a point by Q. If $PQ=r$, the coördinates of Q are

$$x'+r\cos\theta, \quad y'+r\sin\theta.$$

Since Q is on the locus, its coördinates satisfy the equation of the locus. Substituting, and arranging in ascending powers of r, we have

$$0=u_0+ru_1+r^2u_2+\dots\dots\dots \qquad (a)$$

If $u_0=0$, one value of r is zero; that is, one position of Q is coincident with P, or, the point P is on the locus.

Another value of r will be zero and the line l is said to have two points in common with the locus at P, if θ be chosen so that $u_1=0$. The line l is then said to be a *tangent* at P. [Notice that we do not have Q to coincide with P, and then speak of the tangent as a line through two *coincident* points; neither do we have Q approach P, and define the tangent as the limit of the line PQ. We are not troubled with the idea of limits.] The equation of l is then

$$y-y'=(x-x')\tan\theta,$$

where θ has the value which makes $u_1=0$. [If $\theta=90°$, the equation of l is $x-x'=0$.]

EXAMPLE I. Find the equation of the tangent at the point $P\equiv(x', y')$ to the locus $y^2=4ax$. If a line l through P making an angle θ with OX have a point Q in common with the locus its coördinates

$$x'+r\cos\theta, \quad y'+r\sin\theta,$$

where $PQ=r$, satisfy the equation of the locus.

Consequently $0=(y'^2-4ax')+2r(y'\sin\theta-2a\cos\theta)+r^2\sin^2\theta$.

Since P is on the locus,

$$y'^2-4ax'=0. \qquad (1)$$

Hence one value of r is zero for all values of θ. Another value will be zero, that is, the line will be a tangent, if $y'\sin\theta-2a\cos\theta=0$. The equation of the tangent is therefore

$$(y-y')y'=(x-x')2a,$$

or, by using (1), $$yy'=2a(x+x').$$

If $u_0=0$ and $u_1=0$ for all values of θ, then two values of r are zero for any position of l through the point P, which is then called a *double point* on the locus. When this is the case, the line l is called a tangent when θ is chosen so that $u_2=0$, thus making one more value of r zero.

EXAMPLE II. Find the tangent at the origin to the locus $x^2-y^2-3y^3=0$. If a line l through the origin making an angle θ with OX have a point Q in common with the locus, its coördinates,

$$r\cos\theta, \quad r\sin\theta,$$

where $PQ=r$, satisfy the equation.

Thus $r^2(\cos^2\theta-\sin^2\theta)-3r^3\sin^3\theta=0$.

Two values of r are zero for all values of θ. Hence the origin is a double point of the locus. Another value of r will be zero, that is, the line l will be a tangent, if $\cos^2\theta-\sin^2\theta=0$; that is, if $\tan\theta=\pm1$. We get therefore two positions of l satisfying the required condition, its equation being $y=x$ or $y=-x$.

EXAMPLE III. If an equation of the second degree represents a locus with a double point, it is the equation of two straight lines. Denote the equation by

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and assume that it has a double point $P\equiv(x', y')$. Through P draw a line l making an angle θ with OX, and denote any point common to the line and the locus by Q. The coördinates of Q, viz.,

$$x'+r\cos\theta, \quad y'+r\sin\theta,$$

where $PQ=r$, satisfiy the equation of the locus.

Hence $0=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c+2r[(ax'+hy'+g)\cos\theta$
$+(hx'+by'+f)\sin\theta]+r^2[a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta]$.

Since at a double point two values of r must be zero, for all values of θ, we must have

$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0, \tag{1}$$

$$ax'+hy'+g=0, \tag{2}$$

$$hx'+by'+f=0. \tag{3}$$

Multiplying (2) by x', (3) by y', and subtracting their sum from (1), we have

$$gx'+fy'+c=0. \tag{4}$$

Eliminating x', y' from (2), (3), (4), we have

$$\begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix} =0,$$

which is the condition that the given equation represents two straight lines.

EXAMPLE IV. Find the tangent at the origin to the locus $x^2+2y^2+3y^3=0$. Proceeding as in example II, we get

$$r^2(\cos^2\theta+2\sin^2\theta)+3r^3\sin^3\theta=0.$$

Two values of r are zero, so that the origin is a double point on the locus. No value of θ can be found which will make another value of r zero. Hence the locus has no real tangent at the origin. Such a point is called a *conjugate* point, and if we were to trace the curve we would find that we could not find another point satisfying the equation and as near as we pleased to the origin. [While it is true that when there are no tangents at a point on a locus, it is a conjugate point, it does not hold conversely that at a conjugate point tangents are not obtained by the usual methods. For example, we obtain $y=0$ as the tangent at the origin to the locus $y^2=2x^2y+x^4y-2x^4$, although the origin is a conjugate point.]

If $u_0=0$, and the value of θ which makes $u_1=0$ also makes $u_2=0$, but does not make $u_3=0$, the equation gives three zero values of r instead of two for the position of l given by $u_1=0$. Such a point is called a *point of inflexion*.

EXAMPLE V. Find the tangent at the origin to the locus $x^3+x+y=0$. Proceeding as in example II we get

$$0=r(\cos\theta+\sin\theta)+r^3\sin^3\theta.$$

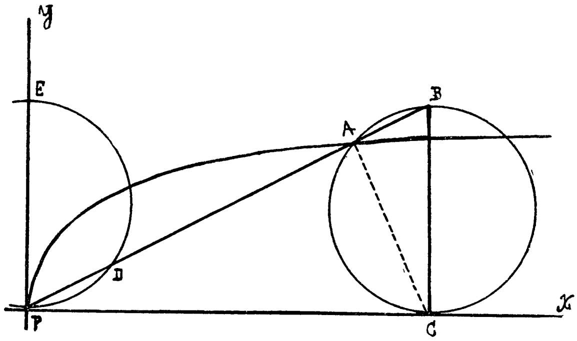
If θ be chosen so that $\cos\theta+\sin\theta=0$, we get *two* additional zero values for r. Hence the origin is a point of inflexion, the tangent at that point being $x+y=0$.

A LINKAGE FOR THE KINEMATIC DESCRIPTION OF A CISSOID.

By JOHN JAMES QUINN, Ph. D.

THEOREM. *If a line pivoted on an axis be drawn to the extremity of the diameter of a constant circle, the locus of its intersection with the circumference of the circle as the latter rolls along the axis is a cissoid.*

PROOF. Let BC be the diameter of the rolling circle, at any point in the axis PX as C; PB the line pivoted at P and intersecting the circle at the point A. Draw EP parallel to BC; then connect E and B. Hence,



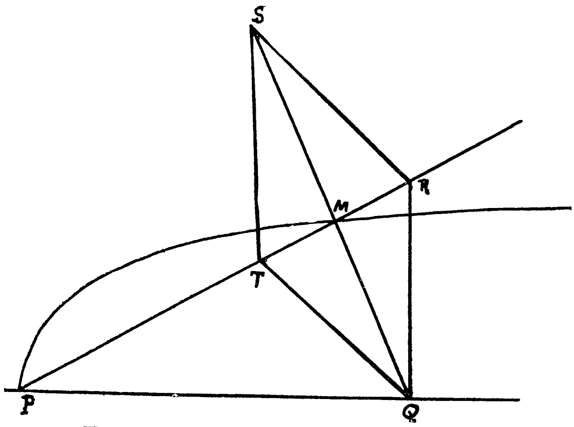
$EB \parallel PC$, $\quad \measuredangle EBA = \measuredangle BPC$,
$PD = AB$.

Hence the locus of A is a cissoid.

COROLLARY. *If from the moving vertex of a rectangle having one side constant, a perpendicular be dropped upon the varying diagonal the locus of the foot of the perpendicular is a cissoid.*

For, in rectangle $EBCP$ the side BC is constant, PB the diagonal, and AC is perpendicular to PB.

THEOREM. *If one diagonal of a rhombus be produced and pivoted at a fixed point in an axis, and if the rhombus be moved along the axis in such a way that one side is constantly perpendicular to it, the locus of the intersection of the diagonals is a cissoid.*



PROOF. Let $QRST$ be the rhombus; RQ perpendicular to PQ; the diagonal TR produced and pivoted at P.

Then since QR is constant, $\angle RMQ$ is a right angle, and the locus of M is a cissoid.

The latter theorem suggests a simple method of constructing a linkage for describing a cissoid by continuous motion.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

254. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\dfrac{n^2}{(16n^2-1)^2}$ beginning with $n=1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\sum_{n=1}^{\infty}\frac{n^2}{(16n^2-1)^2}=\tfrac{1}{64}\Sigma\left[\frac{1}{(4n^2-1)^2}+\frac{1}{4n-1}-\frac{1}{4n+1}+\frac{1}{(4n-1)^2}\right]$$

$$=\tfrac{1}{64}\left[\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}\ldots\ldots\right)+\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}\ldots\ldots\right)\right]$$

$$=\tfrac{1}{64}\left[\left(\frac{\pi^2}{8}-1\right)+\left(1-\frac{\pi}{4}\right)\right]=\tfrac{1}{64}\left(\frac{\pi^2}{8}-\frac{\pi}{4}\right).$$

Also solved by G. W. Greenwood.

255. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Let f be the binary cubic $a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_1^2+a_3x_2^3$, $\triangle=(f,f)_2$ the covariant, the second transvectant of f over itself, and $R=2[4(a_0a_2-a_1^2)\times(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2]=(\triangle,\triangle)_2$=the second transvectant of $\triangle$ over itself. Then if $\triangle_{\kappa\lambda}$ is the $\triangle$ covariant for the cubic pencil $\kappa f+\lambda Q$, Q being the first transvectant of f over $\triangle$ we have $\triangle_{\kappa\lambda}=(\kappa^2-\frac{1}{2}\lambda^2R)\triangle$.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

If $R\neq0$, we may reduce the quantic to the form mX^3+nY^3.
Hence, $\triangle=2mnXY$, $R=-2m^2n^2$, $Q=m^2nX^3-mn^2Y^3$,

$$\kappa f+\lambda Q=(\kappa m+\lambda m^2n)X^3+(\kappa n-\lambda mn^2)Y^3,$$

$$\text{and }\triangle_{\lambda\kappa}=2mn(\kappa^2-\lambda^2m^2n^2)XY=(\kappa^2-\tfrac{1}{2}\lambda^2R)\triangle.$$

If $R=0$, we may reduce the quantic to the form $3lX^2Y$.
Then $\triangle=-2l^2X^2$, $Q=2l^3X^3$, $\kappa f+\lambda Q=3\kappa lX^2Y+2l^3\lambda X^3$.
Hence $\triangle_{\lambda\kappa}=-2\kappa^2l^2X^2=\kappa^2\triangle$.

Also solved by M. E. Graber.

CALCULUS.

215. Proposed by PROFESSOR B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Prove that, if the differential equation $cydx-(y+a+bx)dy-nx(xdy-ydx)=0$, be transformed into an equation between u and x by the substitution $u(y+a+bx+nx^2)=y(c+nx)$, then the variables are separable; and reduce the equation to the form $dv/\phi(v)=dx/\phi(x)$ by the further substitution $v=\alpha u+\beta$, α and β being suitably determined. *Euler*. [Forsyth's *Differential Equations*, p. 48, Ex. 4.]

Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

The first equation may be written

$$\frac{dy}{dx}=\frac{(c+nx)y}{y+a+bx+nx^2}.\quad \text{Thus } \frac{dy}{dx}=u,$$

and as $u(y+a+bx+nx^2)=(c+nx)y$, we have by differentiating with respect to x, writing u for $\frac{dy}{dx}$, and $\frac{u(a+bx+nx^2)}{c+nx-u}$ for y, and re-arranging,

$$\frac{du}{[c^2-bc+na+u(b-2c)+u^2]u}=\frac{dx}{(a+bx+nx^2)(c+nx)}.$$

This is of the form $\frac{du}{f(u)}=\frac{dx}{\phi(x)}$.

Let $u=c+nv$, then $du=ndv$, $f(u)=n(a+bv+nv^2)(c+nv)$. Hence,

$$\frac{dv}{(a+bv+nv^2)(c+nv)}=\frac{dx}{(a+bx+nx^2)(c+nx)}$$

which is of the form $\frac{dv}{\phi(v)}=\frac{dx}{\phi(x)}$.

Also solved by W. W. Landis, and G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

132. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Disregarding the order of λ, μ, ν, how many sets of solutions has the congruence $\lambda+\mu+\nu\equiv0 \pmod{p-1}$ (p prime)? [A. E. Western.]

*Solution by the PROPOSER.

Let n_i be the number of solutions in which i of the numbers λ, μ, ν are equal. If $p\equiv1 \pmod 3$ $n_3=3$, the solutions being

*See problems for solution, *Diophantine Analysis*, No. 134.

$$\lambda \equiv \mu \equiv \nu \equiv 0, \text{ or } \frac{p-1}{3} \text{ or } \frac{2(p-1)}{3} \pmod{p-1}.$$

If $p \equiv 2 \pmod 3$, $n_3=1$, viz., $\lambda \equiv \mu \equiv \nu \equiv 0 \pmod{p-1}$. Next when two are equal so that the congruence is $\lambda+2\mu \equiv 0 \pmod{p-1}$, μ cannot be $\equiv 0, \frac{p-1}{3}$, or $\frac{2(p-1)}{3}$ for then it would be $\equiv \lambda$. With these exceptions μ can have any value, and for each value of μ the congruence gives one value of λ. Hence if $p \equiv 1 \pmod 3$, $n_2=p-4$, and when $p \equiv 2 \pmod 3$, $n_2=p-2$.

If account be taken of the order of λ, μ, ν the total number of solutions of the congruence is $(p-1)^2$ since each unknown may assume $p-1$ values and when two are fixed the other is determined by the congruence. In terms of n_1, n_2 and n_3 the above totality of solutions equals $6n_1+3n_2+n_3$. Hence $6n_1+3n_2+n_3=(p-1)^2$, so that if $p \equiv 1 \pmod 3$, $n_1=\frac{1}{6}(p^2-5p+10)$, and when $p \equiv 2 \pmod 3$, $n_1=\frac{1}{6}(p^2-5p+6)$. Finally, when $p \equiv 1 \pmod 3$, the total number (N) of solutions, disregarding order, of the given congruence is

$$N=3+p-4+\tfrac{1}{6}(p^2-5p+10)=\tfrac{1}{6}(p^2+p+4),$$

and if $p \equiv 2 \pmod 3$,

$$N=1+p-2+\tfrac{1}{6}(p^2-5p+6)=\tfrac{1}{6}(p^2+p).$$

GEOMETRY.

276. Proposed by L. B. NEWCOMB, Los Gatos, Cal.

AF, MN are parallel lines indefinitely extended toward FN; at right angles to AF, MN is AM of length 22; upon the base AB, which is in line with AM, is the triangle ABC whose sides are $AB=21$, $BC=10$, $AC=17$; find the sides of the maximum similar triangle with base extending from B to some point in AF, the vertex in line with MN.

Solution by G. B. M. ZERR. A. M., Ph. D.. Parsons, W. Va.

Let D be the vertex in AF, E the vertex in MN, $AD=x$, $ME=y$. Two cases are possible, BD the short side or BD the long side. The latter gives the maximum. Let $BD=21M$, $DE=10M$, $BE=17M$. Then

$$441M^2=441+x^2 \;\ldots\ldots\ldots\; (1),$$

$$289M^2=1+y^2 \;\ldots\ldots\ldots\; (2),$$

$$100M^2=(x-y)^2+484 \;\ldots\ldots\ldots\; (3).$$

Substitution of x, y from (1) and (2) in (3) gives $\sqrt{[(M^2-1)(289M^2-1)]}=15M^2+1$. Hence $M^2=5$, $M=\sqrt{5}$, and the sides are $21\sqrt{5}$, $17\sqrt{5}$, $10\sqrt{5}$.

Also solved by the Proposer.

280. Proposed by **WILLIAM HOOVER, Ph. D.,** Athens, Ohio.

On any diameter of a given ellipse is taken a point such that the tangents from it intercept on the tangent at one end of the diameter a length equal to the diameter; the ellipse being $a^2y^2+b^2x^2-a^2b^2=0$. Prove that the locus of the point is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$.

Solution by the PROPOSER.

Let the diameter through any point $P(x_1, y_1)$ of the locus cut the ellipse in A, A'. The chord RS of contact of P and the tangent at A' are parallel to the diameter at A_1A_2 conjugate to AA': and then

$$\frac{PB}{RS}=\frac{PA'}{TQ}\text{.........}(1),$$

B being the intersection of RS and PA'. The equation to RS is

$$y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}\text{.........}(2).$$

If $y=mx+c$........(3), is the equation to any chord of $a^2y^2+b^2x^2=a^2b^2$..........(4), the length of chord is $l=2ab\surd[(1+m^2)(m^2a^2+b^2-c^2)]\div(m^2a^2+b^2)$..........(5). Comparing (2) and (3), $m=-\dfrac{b^2x_1}{a^2y_1}$.........(6), $c=\dfrac{b^2}{y_1}$........(7). Substituting (6) and (7) in (5), we have

$$RS=2\surd[(a^4y_1^2+b^4x_1^2)(b^2x_1^2+a^2y_1^2-a^2b^2)]\div(a^2y_1^2+b^2x_1^2)\text{..........}(8).$$

OP or $y=\dfrac{y_1}{x_1}x$........(9) cuts the curve in

$$x'=-\frac{abx_1}{\surd(a^2y_1^2+b^2x_1^2)},\quad y'=-\frac{aby_1}{\surd(a^2y_1^2+b^2x_1^2)}\text{..........}(10).$$

(9) cuts (2) in B, or

$$x''=\frac{abx_1}{\surd(a^2y_1^2+b^2x_1^2)},\quad y''=\frac{aby_1}{\surd(a^2y_1^2+b^2x_1^3)}\text{..........}(11).$$

Then $\overline{PB}^2=(x_1^2+y_1^2)(a^2y_1^2+b^2x_1^2-a^2b^2)^2\div(a^2y_1^2+b^2x_1^2)^2$.........(12),
$\overline{PA'}^2=(x_1^2+y_1^2)(a^2y_1^2+b^2x_1^2+a^2b^2)^2\div(a^2y_1^2+b^2x_1^2)$.........(13),
$\overline{TQ}^2=4a^2b^2(x_1^2+y_1^2)\div(a^2y_1^2+b^2x_1^2)$........ (14).

Squaring (1), substituting (8), (12), (13), and reducing, we have

$$\begin{aligned}&a^2b^2(x_1^2+y_1^2)(a^2y_1^2+b^2x_1^2-a^2b^2)\\&=(a^4y_1^2+b^4x_1^2)(\surd[a^2y_1^2+b^2x_1^2]+ab)^2\text{..........}(15),\end{aligned}$$

the locus required. Equation (15) may be factored by the solution of Algebra problem 250, the relevant factor, with subscripts omitted, being

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

Also solved by G. B. M. Zerr.

281. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

In the proposition in solid geometry "If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane of the lines," it is assumed that two intersecting lines have a common perpendicular. Prove it.

Solution by DR. GEORGE BRUCE HALSTED, Gambier, Ohio.

To prove that two intersecting straights have a common perpendicular.

To each of the given straights, a, b, at their intersection point A construct a perpendicular plane (Halsted's *Rational Geometry*, §337). They have a straight in common (H. R. G. I 6), which is perpendicular to a and perpendicular to b by H. R. G. §333, and that without *assuming* that any two intersecting straights have a common perpendicular.

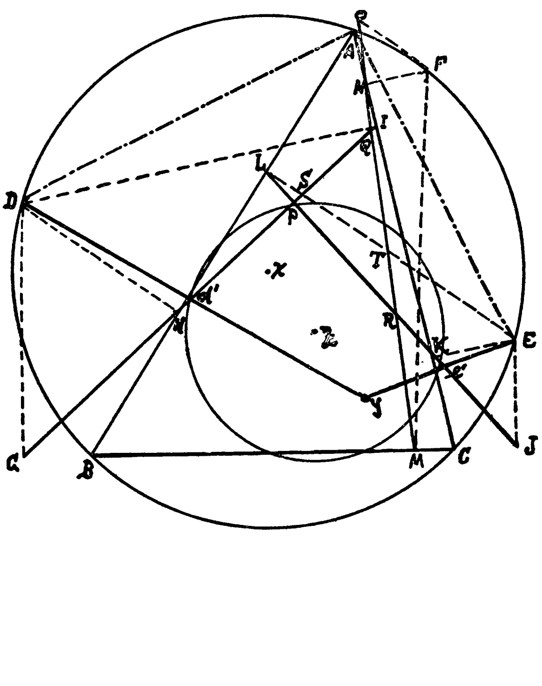
Also solved by A. H. Holmes, Rev. J. H. Meyer, and G. B. M. Zerr.

282. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The pedal lines of any two points on the circum-circle of a triangle concur in an angle equal to that subtended by the said points.

Solution by the PROPOSER.

Upon the circum-circle of triangle ABC take, first, any diameter DE. Then its pedal lines GHI and JKL will concur at P in a right angle.



Join DA, EA, and let GHI cut the perpendicular EL to AB in S. Then EKA and ELA being right angles, $EKLA$ are concyclic about EA; and angle $EAK=ELK$. Similarly, $DHIA$ are concyclic about DA; and $ADI=AHI$. Therefore the right angled triangles AID and SLH are similar; with angles DAI and LSP equal. And hence, even as the angles DAI and EAK are complementary, being within the semi-circle DAE; so also are their equals SLP and LSP. So that LPS is a right angle.

Next, take any other point F upon the circumcircle; and let its pedal line MNO meet those of D and E in Q and R, respectively. Join AF; and let MNO cut the perpendicular EL to AB in T, even as GHI cut it in S.

Then angle DAI has been shown equal to LSP; and hence also to QST. While EL and FO being parallel, QTS is equal to QOF. But $FNAO$ being concyclic about ON, QOF and FAN are equal; while angle DAF is compounded of DAI and FAN. So that GQO, or $QST+QTS$ are equal to DAF; and thus their supplement GQM to the angle subtended by arc DAF.

And lastly; since PRQ is the complement to PQR, even as the angle subtended by arc FE is to that subtended by arc DF; the said angles PRQ and that subtended by arc FE must be also equal. And similarly for any point upon the circum-circle.

Corollary. DE being a diameter of the circum-circle, the medial points,—say u, v, and w,—of the sides AB, BC, and CA of the inscribed triangle must also bisect the segments LH, GJ, and KI cut off upon said sides by its pedal lines GHI and JKL. Each bisecting perpendicular,—say xu, xv, and xw,—from x, the circum-center, being parallel to DH and EL, DG and EJ, and DI and EK, respectively. And thus the said medial points u, v, and w are always the circum-centers of the right angled triangles HPL, GPJ, and KPI, respectively.

Also solved by G. W. Greenwood, and J. Scheffer.

283. Proposed by REV. ALAN S. HAWKESWORTH, Allegheny, Pa.

The right angled intersection of the pedal lines of any diameter of the circumcircle lies on the "nine points circle" of the inscribed triangle.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

Let x [see figure above] be the circum-center, y the ortho-center, and z the center of the nine points circle of triangle ABC; passing, by construction, through the feet of the perpendiculars from the ortho-center upon the sides of the triangle, and also through the medial points of the lines joining the ortho-center to the vertices of the triangle.

Then, by known theorems, the center of z of the nine points circle bisects xy, the distance between the ortho- and circum-centers. While the said lines from the orth-ocenter perpendicular to the sides of the triangle cut its circum-circle at twice their distance to these sides. So that the ortho-center Y is evidently also a center of similitude to the nine points and circum-circles; with all magnitudes of the latter half those of the former.

And hence d', where DY, by a known theorem, is bisected by the pedal line GHI of D, is also its intersection by the nine points circle. And similarly, e', the middle point of YE, lies both on the pedal line JKL of E, and on the nine points circle.

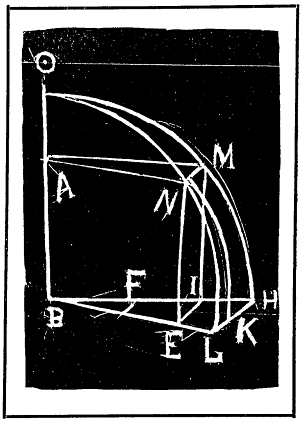
Therefore $d'e'$ is parallel to DE, and bisects XY in Z, and is thus a diameter of the nine points circle; even as DXE is of the circum-circle. So that $d'Pe'$ being a right angle, P lies on the nine points circle.

273. Proposed by A. H. HOLMES, Brunswick, Maine.

Required a purely geometrical solution of the problem, to find the contents of a solid generated by the revolution of a semi-segment of a circle about the sine of its arc.

Solution by the PROPOSER.

HBO is a quadrant whose revolution about *BO* as an axis generates a hemisphere. *BAF* is a semi-segment of radius$=BO$. Draw *AM* parallel to *HB* and *MI* parallel to *BO*. Suppose the quadrant to revolve about its axis a very small distance, the point *H* moving to *L* so as to generate *HBOLB*, *M* falling on *N*. Through *NE* pass a plane parallel to *HBO*. The semi-segment $HMI=AFB$ generates *HIELMN*; of which the part $EKLN=$part generated by *BAF*.



It is obvious that the volume generated by the semi-segment *BFA* in an entire revolution will equal that generated by *HMI* *minus* the sum of the solids *HIKEMN* lying about the circumference of the base of the hemisphere.

But $MNHIEK=IE\times$area of the semi-segment and the entire sum of all these is equal to the circumference described by *BI* as radius into the same area. If we put $BO=r$, $BE=c$, $BA=s$, and arc $AF=a$, we obtain for the solid generated by *BOMI*, $\frac{2\pi}{3}(sc^2+r^3-r^2s)$. Consequently, for the solid generated by *HIM*, $\frac{2\pi}{3}(r^2s-sc^2)$. The sum of all the solids $HKEINM=$semi-segment $MHI\times 2\pi c$ $=\pi(car-sc^2)$. Consequently the volume sought is

$$=\frac{2\pi}{3}(r^2s-sc^2)-\pi(car-sc^2)=\pi(\frac{sc^2}{3}+\frac{2r^2s}{3}-car).$$

Putting $c^2=r^2-s^2$, this becomes $\pi(sr^2-s^3/3-car)$.

GROUP THEORY.

14. Proposed by O. E. GLENN, Springfield, Mo.

Hölder has proved* that any group (G) of order $\sum\limits_{i=1}^{n} p_i$ (p_i a prime$\neq p_j$) may be generated as follows: $M^\mu=N^\nu=1$, $N^{-1}MN=M^\alpha$, where $\{M\}$ is the product of all the invariant subgroups of G of prime order and $\{N\}$ is any one of a set of conjugate cyclical subgroups of order ν, $(\sum\limits_{i=1}^{n} p_i=\mu\nu)$. Find the generating relations of G in terms of operations of prime order, and express M and N in terms of these operations, for $n=4$.

*See Burnside, *Theory of Groups*, p. 353.

PROBLEMS FOR SOLUTION.

ALGEBRA.

259. Proposed by ARTEMUS MARTIN, M. A., Ph. D., LL. D., Washington, D. C.

On page 167 of George Bruce Halsted's *Metrical Geometry* (Mensuration), Boston, 1881, Table of Scalene Triangles, is found the following triangle, viz., Sides 21, 61, 65; Area 420. The sides of a rational scalene triangle, whose sides have no common divisor, can not all be odd; one must be even and the other two odd. It is required to find the error in the sides of the above triangle, assuming that the area is correct.

260. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The necessary and sufficient condition that a binary form be a perfect nth power is that its Hessian vanish.

261. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series, $\frac{1}{n^p}+\frac{3}{n^{2p}}+\frac{5}{n^{3p}}+\frac{7}{n^{4p}}+\frac{9}{n^{5p}}+\dots\dots\dots$

AVERAGE AND PROBABILITY.

176. Proposed by T. N. HAUN, Mohawk, Tenn.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? (Problem 72, p. 503, Williamson's *Integral Calculus.*)

CALCULUS.

217. Proposed by PROFESSOR F. ANDEREGG, Oberlin College, Oberlin, Ohio.

Find $\lim\limits_{n \doteq \infty} \frac{1}{n}\sqrt[n]{[(n+1)(n+2)\dots\dots(2n)}$.

218. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin mx \sin nx}{\sin x}dx$; (b) $\int_0^{\frac{1}{2}\pi}\frac{\cos mx \sin nx}{\sin x}dx$, where n is a positive integer. Also, modify the result for the case of m an integer.

219. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

In the article "Infinitesimal Calculus" in the *Encyclopaedia Britannica* Vol. XIII, page 24, I notice the following: "Of all triangular pyramids standing on a given triangular base, and of given altitude, find that whose surface is the least." A solution is required.

DIOPHANTINE ANALYSIS.

134. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

How many sets of solutions has the congruence $\lambda+\mu+\nu+\xi\equiv0 \pmod{p-1}$ p being a prime number; the order of λ, μ, ν, ξ being disregarded.

GEOMETRY.

284. Proposed by JOHN JAMES QUINN, Ph. D., Warren, Pa.

a) Suppose that two radii R and r, whose center is the origin, revolve with uniform angular velocities 3θ and θ, respectively. What is the equation of the locus of P, the projection parallel to the X axis of the extremity of the radius r on the radius R produced if necessary.

b) Apply this curve to the trisection of an angle.

c) Suppose the ratio of their velocities is $n\theta:\theta$. Show how we can effect the multisection of an angle.

285. Proposed by G. E. BROCKWAY, Nashua, N. H.

Prove without the aid of the circle, that if the bisectors of the angles of a triangle be drawn, the greatest bisector falls on the least side.

286. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

On the sides of a given triangle measure off equal distances from the extremities of the base, and at these points erect perpendiculars to the sides. Find the locus of the point of intersection of these perpendiculars. Solve by methods of analytic geometry.

287. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Show that the points whose abscissae are 0, $a\sqrt{3}$, and $-a\sqrt{3}$ are points of inflexion on the locus $x^2y-a^2x+a^2y=0$.

MISCELLANEOUS.

157. Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in the Educational Times, London.)

An inelastic rod 9 feet long is placed with its upper end upon a rough vertical plane and its lower end on a smooth horizontal plane, and so that it makes an angle of 45° with each plane. It is now released, and strikes against a smooth sphere of 1 foot diameter placed in contact with the two planes. Determine the subsequent motion.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Group Theory, 8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the

eight players being paired in each round, every pair to be matched once and but once in the tournament. List the possible programs different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

Miscellaneous, 151. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Sum the series $\sum_{r=1}^{r=m} \operatorname{cosec}\left[\frac{2r-1}{4m}\pi+\theta\right]\operatorname{cosec}\left[\frac{2r-1}{4m}\pi-\theta\right]$.

NOTES AND NEWS.

Mr. George Brett has been elected tutor in mathematics in the College of the City of New York.

The roll of foreign members of the Circolo Mathematico Di Palermo includes the names of forty-five Americans.

The list of active members just issued by the London Mathematical Society contains the names of thirty American mathematicians.

Father J. G. Hagen, S. J., professor of astronomy in Georgetown University, has been offered the directorship of the Vatican Observatory.

At its December meeting the Chicago Section of the American Mathematical Society elected officers for the year 1906, as follows: Chairman, Professor Alexander Ziwet; Secretary, Professor H. E. Slaught; additional member of the Executive Committee, Professor A. G. Hall.

Probably the oldest journal devoted to *elementary* mathematics is the *Maandelykse Mathematische Liefhebbery*, 1754—1769. The John Crerar Library in Chicago, has the complete set of seventeen volumes of this journal, which was published at Purmerende in the Netherlands.

Professor James Mills Pierce, Perkins professor of mathematics and astronomy at Harvard University, has presented his resignation to take effect a year hence. Professor Pierce's service at Harvard covers a period of fifty-two years, he having been appointed tutor in the University in 1854.

Dr. J. J. Quinn of Warren, Pa., is preparing a monograph on the trisection of the angle. It is to contain a large variety of solutions of the historic problem by higher plane curves and by analytic methods, with historical notes. He desires that all who have suggestions as to methods of dealing with the problem should send them to him. For these suggestions proper credit will be given.

The current number of the *Proceedings of the London Mathematical Society* contains the following contributions: "On the arithmetical nature of the coefficients in a group of linear substitutions of finite order" (second paper), by Pro-

fessor W. Burnside; "The continuum and the second number class," by Mr. G. H. Hardy; "On 'well-ordered' aggregates," by Professor A. C. Dixon; "On the arithmetic continuum," by Dr. E. W. Hobson; "On some difficulties in the theory of transfinite numbers and order types," by Mr. B. Russell; "On the Hessian configuration and its connection with the group of 360 plane collineations," by Professor W. Burnside; "On the representation of certain asymptotic series as convergent continued fractions," by Professor L. J. Rogers.

Dr. Hendrik Antoon Lorentz, professor of mathematical physics in the University of Leiden, has begun a course of lectures at Columbia University on the theory of electrons and it applications to the phenomena of light and radiant heat. The program is as follows:

Friday, March 23, 4 to 6 p. m.; Saturday, March 24, 10 to 12 a. m.; and Friday, March 30, 4 to 6 p. m.—General principles; theory of free electrons.

Saturday, March 31, 10 to 12 a. m., and Friday, April 6, 4 to 6 p. m.—Emission and absorption of heat.

Saturday, April 7, 10 to 12 a. m.; Wednesday, April 11, 4 to 6 p. m.; and Thursday, April 12, 4 to 6 p. m.—The Zeeman effect. Propagation of light in ponderable bodies.

Thursday, April 26, 4 to 6 p. m., and Friday, April 27, 4 to 6 p. m.—Optical phenomena in moving systems.

The following were elected members of the American Mathematical Society at its February meeting in New York: Mr. M. J. Babb, University of Pennsylvania; Mr. William Betz, East High School, Rochester, N. Y.; Mr. G. D. Birkhoff, University of Chicago; Mr. W. D. Breuke, Harvard University; Mr. B. E. Carter, Massachusetts Institute of Technology; Dr. H. L. Coar, University of Illinois; Miss Anna Johnson, Harvard University; Mr. W. D. Lambert, U. S. Coast Survey; Mr. W. A. Luby, Central High School, Kansas City, Mo.; President W. J. Milne, New York State Normal College; Professor Richard Morris, Rutgers College; Mr. W. J. Newlin, Harvard University; Miss R. A. Pesta, Wendell Phillips High School, Chicago, Ill.; Dr. H. B. Phillips, University of Cincinnati; Mr. A. R. Schweitzer, University of Chicago; Mr. C. G. Simpson, Michigan College of Mines; Mr. A. W. Stamper, Columbia University; Mr. F. C. Touton, Central High School, Kansas City, Mo.; Mr. M. O. Tripp, College of the City of New York.

The following periodicals have been received: The Scientific American, The Educational Times, The Nation, The Review of Reviews, The Literary Digest, Ohio Educational Monthly, The Ohio Teacher, The Physical Review Bulletin of American Mathematical Society, School Science and Mathematics, Proceedings of the London Mathematical Society, The Open Court, The School Visitor, Popular Astronomy, L'Enseignement Mathematique, Bollettino della Associazione, Bob Taylor's Magazine, The University Herald.

For the Metric System The following is the text of a bill which has recently been introduced by Representative Littauer to fix the standard of weights and measures in the United States by the adoption of the metric system:

Be it enacted by the Senate and House of Representatives of the United States of America in Congress assembled:

That from and after the first of July, nineteen hundred and eight, all of the Departments of the Government of the United States, in the transaction of business requiring the use of weight and measurement, shall employ and use the weights and measures of the metric system.

The committee of publicity of the American Metrological Society, of which Professor Simon Newcomb is chairman, has sent out a circular letter in support of the bill. Some quotations follow:

"Notwithstanding some recent misleading statements to the contrary, made by opponents of the bill, the Metric System during the past thirty years has made the most substantial and important progress of its history. By the establishment of the International Bureau of Weights and Measures in 1872, the Metric System became in the fullest sense an International System. Its subsequent introduction into actual and general use in Germany and the neighboring countries have given it the character of a real International System, and secured for it a commanding position which neither the British nor any other system ever possessed, and which make it as near a permanent institution as any human arrangement can be. At the same time it is among English speaking people themselves, the medium in which all scientific research is carried on, the system in which all electrical measurements are made, and in which all higher education is given, for which reason thousands of our young people are already acquainted with it.

Under present conditions the British system is an ugly excresence on the world's literature and practical arts which the general welfare requires we should abolish as speedily as possible. Already the conflict of two systems is a seriou obstacle to international trade and a hindrance to international coöperation in every direction. The sentiment in favor of the Metric System is so far advanced in the British Empire that it is a question whether we will not be anticipated in its adoption. The expressions of Boards of Trade, educational bodies, and Colonial Governments leave no doubt but that England would immediately follow us in the adoption of the Metric System should we be fortunate enough to first take the step.

For these reasons, among others, we earnestly request you to obtain the largest possible expression of opinion favorable to the introduction of the system into all Government work by Act of Congress."

Errata.

Page 27, line 11 from bottom, for 15 read 13.

Page 18, lines 5 and 6 from bottom. The result of substituting in Lagrange's formula should read,

$$v=\frac{1}{x}\left[1+\frac{\log_e x}{x}+\frac{3}{2!}\left(\frac{\log_e x}{x}\right)^2+\cdots\cdots+\frac{n^{n-2}}{(n-1)!}\left(\frac{\log_e x}{x}\right)^{n-1}+\cdots\cdots\cdots\right]$$

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. APRIL, 1906. No. 4.

A GENERALIZED TRIGONOMETRIC SOLUTION OF THE CUBIC EQUATION.

By W. D. LAMBERT, Coast and Geodetic Survey.

The trigonometric solution of the cubic, as commonly given, is either limited to the "irreducible case," or treated by a special method for each case. The following solution treats all cases by the same device, and if tables of hyperbolic functions are at hand, is no less convenient for numerical calculation than other methods. Nor is it restricted to real coefficients, a fact which may sometimes make it convenient in applications of the theory of functions.

As a preliminary, consider the sine, $p+iq$, belonging to a complex angle $u+iv$, *i. e.*, let

$$\sin(u+iv)=p+iq=re^{ai}. \tag{1}$$

By expanding, and separating real and imaginary parts, we find that

$$\sin u \operatorname{Cosh} v=p, \quad \cos u \operatorname{Sinh} v=q. \tag{2}$$

Hence,

$$\frac{p}{\sin u}=\operatorname{Cosh} v, \quad \frac{q}{\cos u}=\operatorname{Sinh} v. \tag{3}$$

Squaring each equation in (2), and subtracting, we find that

$$\frac{p^2}{\sin^2 u}-\frac{q^2}{\cos^2 u}=\operatorname{Cosh}^2 v-\operatorname{Sinh}^2 v=1. \tag{4}$$

By substituting $\cos^2 u=1-\sin^2 u$ in (4), and solving the resulting equation as a quadratic in $\sin^2 u$, we get

$$\left.\begin{aligned} u &= \sin^{-1}\left[\pm\sqrt{\tfrac{1}{2}(p^2+q^2+1)-\tfrac{1}{2}\sqrt{(p^2+q^2+1)^2-4p^2}}\right] \\ &= \sin^{-1}\left[\pm\sqrt{\tfrac{1}{2}(r^2+1)-\tfrac{1}{2}\sqrt{r^4-2r^2\cos\alpha+1}}\right] \end{aligned}\right\} \quad (5)$$

By a similar process'

$$\left.\begin{aligned} v &= \mathrm{Sinh}^{-1}\left[\pm\sqrt{\tfrac{1}{2}(p^2+q^2-1)+\tfrac{1}{2}\sqrt{(p^2+q^2-1)^2+4q^2}}\right] \\ &= \mathrm{Sinh}^{-1}\left[\pm\sqrt{\tfrac{1}{2}(r^2-1)+\tfrac{1}{2}\sqrt{r^4-2r^2\cos2\alpha+1}}\right] \end{aligned}\right\} \quad (6)$$

The signs before the inner radicals are apparently ambiguous; but by considering that $\sin^2 u$ and $\sin^2 v$ must be positive, and $\sin^2 u \leq 1$ (in order to give real values to u and v), we see that the signs of the inner radicals must be as written in (5) and (6). The signs before the outer radicals may be taken the same as those of p and q, respectively.

Suppose the cubic to have been deprived of its second term, and to be in the form

$$x^3-ax+b=0, \tag{7}$$

where a and b may be positive, negative, or complex. From trigonometry,

$$\sin^3\phi-\tfrac{3}{4}\sin\phi+\tfrac{1}{4}\sin3\phi=0. \tag{8}$$

This suggests reducing (7) to the form

$$y^3-\tfrac{3}{4}y+c=0, \tag{9}$$

which is accomplished by the substitution

$$x=2\sqrt{\frac{a}{3}}y, \text{ giving } c=\frac{3b}{8a}\sqrt{\frac{3}{a}}. \tag{10}$$

If a be complex, either determination may be taken for $\sqrt{\frac{a}{3}}$, but $\sqrt{\frac{3}{a}}$ should then have an argument equal to the negative of the argument of $\sqrt{\frac{a}{3}}$.

If we assume $\sin\phi=y$, we find by comparing (8) and (9) that

$$\sin3\phi=\frac{3b}{2a}\sqrt{\frac{3}{a}}. \tag{11}$$

To solve equation (7), we compute $\frac{3b}{2a}\sqrt{\frac{3}{a}}$, treat it as a sine, and compute the corresponding angle—which we call $3\phi_1$—by (5) and (6), taking the real part of 3ϕ between $-90°$ and $+90°$. Other possible values of 3ϕ are $3\phi_2=3\phi_1+360°$, $3\phi_3=3\phi_1+720°$. The roots are therefore, by (10),

$$\left.\begin{aligned} x_1&=2\sqrt{\frac{a}{3}}\sin\phi_1 \\ x_2&=2\sqrt{\frac{a}{3}}\sin\phi_2=2\sqrt{\frac{a}{3}}\sin(\phi_1+120°) \\ x_3&=2\sqrt{\frac{a}{3}}\sin\phi_3=2\sqrt{\frac{a}{3}}\sin(\phi_1+240°) \end{aligned}\right\} \tag{12}$$

When a and b are restricted to real values, we readily find the well-known conditions for the reality of all roots:

$$a>0 \text{ and } 27\,b^2<4a^3.$$

The condition for equal roots holds whether the coefficients are real or complex. By introducing the Gudermannian angle θ, defined by $\phi=\log_e\tan(45°+\frac{1}{2}\theta)$, from which follows that

$$\text{Cosh}\phi=\sec\theta,\quad \text{Sinh}\phi=\tan\theta,\quad \text{etc.,}$$

we can reduce the general formulas given above to the special ones involving tangents of auxiliary angles sometimes used in the case of a single real root.

It is worth noting that a table of Mercator's parts—calculated for a spherical earth—is a table of inverse Gudermannians, and may be used to give the hyperbolic functions if a regular table of them is not at hand. A table of Mercator's parts carried to $\frac{1}{100}$ minute is given in Callet's "Tables Nautiques." The second alternative form under (5) and (6) is generally the more convenient for numerical work with complex coefficients. The inner radical represents the side of a triangle whose other sides are r^2 and 1, and their included angle α. There are small tables for the solution of this case, but in view of its frequent occurrence either directly, or in an equivalent form, it seems rather remarkable that no extensive tables for it are in general use.

Example: Solve $x^3+ix+1+i=0$.

Here $a=-i$, $b=1+i$, $\sqrt{\frac{a}{3}}=\frac{1-i}{\sqrt{6}}$, $\sqrt{\frac{3}{a}}=(1+i)\sqrt{\frac{3}{2}}$, $\sin3\phi=-\frac{3}{2}\sqrt{6}$.

From (5) and (6), since $p=-\frac{3}{2}\sqrt{6}$ and $q=0$, $u=\sin^{-1}(-1)=-90°$, $v=\text{Sinh}^{-1}(\frac{5}{2}\sqrt{2})=1.97544$, $\text{Sinh}\frac{v}{3}=0.70709$, $\text{Cosh}\frac{v}{3}=1.22473$;

$$x_1=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times 1.22473+i\frac{\sqrt{3}}{2}\times 0.70709)=-0.00001+i\times 0.99997,$$

$$x_2=\frac{2(1-i)}{\sqrt{6}}(1.22473)=0.99998(1-i),$$

$$x_3=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times 1.22473-i\frac{\sqrt{3}}{2}\times 0.70709)=-0.99997+i\times 0.00001.$$

These are from five-figure tables; the exact roots are i, $1-i$, and -1.

ON THE EXPANSION OF DEVERTEBRATED THREE DIMENSIONAL DETERMINANTS AND THE EXTENSION OF CAYLEY'S EXPANSION THEOREM.

By ORLANDO S. STETSON, Syracuse University.

The primary object of this paper is to extend to three dimensional or cubical determinants* the general law for the expansion of a two dimensional devertebrated determinant† in terms of the elements of the principal diagonal of the given determinant and their co-axial minors or, in other words, to develop a general law for the expansion of a devertebrated cubical determinant in terms of the elements of its principal diagonal plane and their co-planar minors.

The secondary object is to show how Cayley's Expansion Theorem may be extended to cubical determinants with but slight modifications. A formula for the number of terms of a cubical determinant which are independent of the elements of the principal diagonal plane is also given.

Let $\triangle$ denote a cubical determinant of order n and let $\triangle'_a$ denote a new cubical determinant formed by adding a different variable to each of the n^2 elements of the principal diagonal plane.

Expanding $\triangle'_a$ as a cubical determinent with binomial elements, the term independent of the variable will be the given cubical determinant $\triangle$. The next n^2 terms are the products of the respective variables and their corresponding cubical minors of order $n-1$; similarly, the next $\frac{n^2(n-1)^2}{2!}$ terms are products

*An exposition and bibliography of cubical determinants is given by E. R. Hedrick, *Annals of Mathematics*, Ser. 2, Vol. 1 (1900), pp. 49—67.

†Stetson, MONTHLY, Vol. XI (1904), pp. 166—168.

of pairs of variables of the principal diagonal plane and their corresponding cubical minors of order $n-2$, no two of the variables being common to the same plane, and so on.

Now, let us assume any a elements of the principal diagonal plane of the given cubical determinant to be equal to zero and let us replace each of the corresponding variables by the negative of the element, placing the remaining n^2-a variables equal to zero; the general law may then be stated as follows:

A devertebrated cubical determinant of order n containing a zero elements in its principal diagonal plane may be developed into a series of terms of which the first is the given cubical determinant, $\triangle$; the next a being the products of each of the a elements into its co-planar cubical minor of order $n-1$; the next set consisting of the products of every possible pair of the a elements of the principal diagonal plane and its coplanar cubical minor of order $n-2$, no two of the elements being common to a same plane, and so on; the signs of the respective terms are positive or negative according as the combinations of the a elements are even or odd.

By a proper selection of the a elements this general law may be seen to contain as a particular case the law for the expansion of a devertebrated cubical determinant in which the a zero elements lie along the principal diagonal of the ordinary determinant of the principal diagonal plane.

For $a=n^2$, the expansion will be seen to contain, also, the law for the expression of an invertebrate coplanar minor in terms of the vertebrate coplanar minors and the elements of the principal diagonal plane.

As illustrations, we may assume a cubical determinant, $\triangle'_a$ of order 3 in which the elements in parentheses designate the variables which are added to the elements of the principal diagonal plane. We will write the sheets under one another, thus—

$$\triangle'_a = \left\{ \begin{array}{l} \left\{ \begin{array}{lll} 111+(111) & 112 & 113 \\ 121 & 122+(122) & 123 \\ 131 & 132 & 133+(133) \end{array} \right. \\ \\ \qquad \left\{ \begin{array}{lll} 211+(211) & 212 & 213 \\ 221 & 222+(222) & 223 \\ 231 & 232 & 233+(233) \end{array} \right. \\ \\ \qquad\qquad \left\{ \begin{array}{lll} 311+(311) & 312 & 313 \\ 321 & 322+(322) & 323 \\ 331 & 332 & 333+(333) \end{array} \right. \end{array} \right.$$

Expanding $\triangle'_a$ as a cubical determinant with binomial elements, the expansion may be written in the following form—

$$\begin{aligned} \triangle'_a = \triangle & +(111)[111]+(122)[122]+(133)[133]+(211)[211]+(222)[222] \\ & +(233)[233]+(311)[311]+(322)[322]+(333)[333] \\ & +(111)(222)[111,\ 222]+(111)(322)[111,\ 322]+(211)(122) \end{aligned}$$

$$[211, 122]+(211)(322)[211, 322]+(311)(122)[311, 122]$$
$$+(311)(222)[311, 222]+(122)(233)[122, 233]+(122)(333)$$
$$[122, 333]+(222)(333)[222, 333]+(222)(133)[222, 133]$$
$$+(322)(133)[322, 133]+(322)(233)[322, 233]+(133)(211)$$
$$[133, 211]+(133)(311)[133, 311]+(233)(111)[233, 111]$$
$$+(233)(311)[233, 311]+(333)(111)[333, 111]+(333)(211)$$
$$[333, 211]+(111)(222)(333)+(111)(322)(233)$$
$$+(122)(211)(333)+(122)(233)(311)+(133)(211)(322)$$
$$+(133)(222)(311).$$

For $a=1$, assume $(111)=-111$, and place the remaining variables equal to zero, then

$$\triangle'_1=\triangle-111[111].$$

For $a=2$, assume $(111)=-111$, $(333)=-333$ and place the remaining variables equal to zero, then

$$\triangle'_2=\triangle-111[111]-333[333]+111.222.333.$$

For $a=3$, assume $(111)=-111$, $(211)=-(222)=-222$, and place the remaining variables equal to zero, then

$$\triangle'_3=\triangle-111[111]-211[211]-222[222]+111.222.333.$$

We may now obtain very readily a formula for the number of terms in a cubical determinant which are independent of the elements of the principal diagonal plane.

In the preceding article it has been shown that the expansion of a devertebrated determinant may be put in the following form—

$$\begin{aligned}\triangle_a = \triangle &+ \Sigma(111)[111]+\Sigma(111)(222)[111, 222]\\ &+\Sigma(111)(222)(333)[111, 222, 333]+\dots\dots\\ &+\Sigma(111)(222)(333)\dots\dots aaa[111, 222, \dots\dots, aaa]\\ &+(111)(222)\dots\dots(nnn),\end{aligned}$$

no two elements of a product belonging to a same plane. Placing $a=n$, thereby making $\triangle_n$ an invertebrate cubical determinant, we have

$$\begin{aligned}\triangle_n=\triangle &-\Sigma 111[111]+\Sigma 111.222[111, 222]-\Sigma 111.222.333[111, 222, 333]\\ &+\dots\dots+(-1)^r\Sigma 111.222\dots\dots rrr[111, 222\dots\dots rrr]\\ &+\dots\dots(-1)^n 111.222.333\dots\dots nnn.\end{aligned}$$

Since there are $(n!)^2$ terms in the expansion of a cubical determinant of order n, the number of terms in the preceding invertebrate cubical determinant will be

$$\phi(n)=(n!)^2-n^2[(n-1)!]^2+\frac{n^2\,(n-1)^2}{1.2}[(n-2)!]+\ldots\ldots\ldots$$

$$-\frac{n^2(n-1)^2(n-2)^2}{1.2.3}[(n-3)!]+\ldots\ldots\ldots$$

$$+(-1)^r\frac{n^2(n-1)^2\ldots\ldots\ldots(n-r+1)^2}{1.2.3\ldots\ldots\ldots r}[(n-r)!]+\ldots\ldots\ldots$$

$$+(-1)^n\frac{n^2(n-1)^2\ldots\ldots\ldots 3.2.1}{1.2.3\ldots\ldots\ldots n}.$$

Removing from each of the terms the common factor $[n!]^2$ and noticing that the first two terms are equal but of opposite sign, we have

$$\phi(n)=(n!)^2\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots\ldots(-1)^n\frac{1}{1.2.3\ldots\ldots n}\right\}.$$

CAYLEY'S EXPANSION THEOREM.

Any cubical determinant may be developed into a series of terms of which the first is obtained by changing into zero all the elements of the principal diagonal plane, the next n^2 by multiplying each of the elements of the principal diagonal plane by its minor in the cubical determinant and altering the principal diagonal plane of the minor as the given cubical determinant was altered; the next $\frac{n^2\,(n-1)^2}{2!}$ by multiplying each pair of elements of the principal diagonal plane, no two coming from a same set of planes, by its minor in the cubical determinant and altering the principal diagonal plane of the minor as the original cubical determinant was altered, etc., the last $n!$ terms being every possible product, n at a time, of the elements of the principal diagonal plane, no two being taken from a same set.

By changing into zero all the elements of the principal diagonal plane we delete exactly those terms of the cubical determinant which contain any of these elements. The product of an element of the principal diagonal plane and its minor in the given cubical determinant gives exactly those terms of the given cubical determinant which contain that element.

Changing into zero all the elements of the principal diagonal plane of the minor we obtain the sum of all the terms of the cubical determinants which contain that element and *only* that element.

The expansion gives, therefore, first, all the terms of the determinant involving no element of the principal diagonal plane of the given cubical determinant; secondly, all those involving *only* one element; thirdly, all those involving *only* two elements, etc.

The number of terms in the expansion which contain only one element will be the same as the number of elements in the principal diagonal plane, or n^2.

Since in forming the pair of elements one and only one element of a pair can be taken from the same set of planes, it follows that $n(n-1)$ pairs can be taken from a same two sets of planes in which a corresponding pair of planes in the set are kept fixed and, therefore, in every possible way there can be formed

$$n(n-1)n_2 \text{ or } \frac{n^2\ (n-1)^2}{2\,!}$$

by choosing in every possible manner the pair of planes to be kept fixed.

It should be noticed that the expansion passes directly from those terms involving only $n-2$ elements of the principal diagonal plane to those involving them all. There would be no difficulty in stating at once the general law for the expansion of a cubical determinant in terms of the elements of the principal diagonal of the principal diagonal plane of the cubical determinant and their coaxial cubical minors.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

256. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Three men, A, B, and C, rented a pasture for a fixed amount, each to pay per month in proportion to the stock pastured. During the first month A put in 3 horses and B and C each some horses, and B paid for the month \$6, but A and C each defaulted payment. During the next month each put in one more horse, and C paid for the month \$7.20, but A and B each defaulted payment. During the next month each put in one more horse, and A paid his bill for the month, \$5, but B and C each defaulted.

Required: (1) the rent of the pasture per month; (2) the number of horses B and C each put in during the first month; and (3) the amount A, B, and C, each, owed for the unpaid service.

I. Solution by S. A. COREY, Hiteman, Iowa.

Let a, b, c=rate per horse per month for the first, second, and third month, respectively. Let x, y=number of horses put in the first month by B, and C, respectively. Let $n=3+x+y$=total number of horses in pasture first month. Let m=fixed monthly rental of pasture.

Then as A paid \$5 for the third month's rental when he had 5 horses in the pasture, c=\$1. As fewer horses were in the pasture the two preceding months the rate per horse per month was more than \$1 for the first and second months,

or both a and $b>\$1$. Then as x is an integer, as $a>\$1$, and as B paid but \$6 for the first month's service, $x \gtreqless 5$. Similarly, as y is an integer, $b>\$1$, and as C paid but \$7.20 for the second month's service $(y+1) \gtreqless 7$, or $y \gtreqless 6$. Hence as $n=3+x+y$, $n \gtreqless 14$. But as $c=m/(n+6)=1$, $m=(n+6)$, whence $m \gtreqless \$20$. But as not less than 5 horses were placed in the pasture the first month, $m= \gtreqless 11$.

B had x horses in the pasture the first month, the rate then being m/n, or $m/(m-6)$, and paid \$6 for the service. Hence, $xm/(m-6)=36$, or reducing and transposing, $(6-x)m=36$, but as $(6-x)$=integer, and $11 \gtreqless m \gtreqless 20$ (remembering that C had some horses in the pasture the first month, and that $n=m-6$). $6-x=2$, $m=\$18$, whence $x=4$ and $n=12$. But $y=n-3-x=5$. As $a=m/n=\$1.50$, $b=m/(n+3)=\$1.20$, and $c=\$1$, we readily find that
A owed \$4.50 for the first, and \$4.80 for the second month's service=\$9.30 total.
B owed \$6 for the second, and \$6 for the third month's service=\$12 total.
C owed \$7.50 for the first, and \$7 for the third month's service=\$14.50 total.

II. Solution by REV. J. H. MEYER, S. J., Professor of Mathematics, College of the Sacred Heart, Augusta, Ga.; by M. R. BECK, Cleveland, Ohio, and by J. EDWARD SANDERS, Reinersville, Ohio.

Using r as the number of dollars in one month's rent; x and y as the number of horses B and C put in at first, respectively, we obtain

$$\frac{rx}{3+x+y}=6, \text{ or } r=\frac{18+6x+6y}{x}. \tag{1}$$

$$\frac{r(y+1)}{6+x+y}=7.20, \text{ or } r=\frac{43.2+7.2x+7.2y}{y+1}. \tag{2}$$

$$\frac{5r}{9+x+y}=5, \text{ or } r=\frac{45+5x+5y}{5}. \tag{3}$$

Equating (1) and (3), $90-15x+30y=5x^2+5xy$.........(4). Equating (2) and (3), $171+31x-14y=5xy+5y^2$..........(5). Adding (4) and (5), $-5(x+y)^2+16(x+y)=-261$..........(6). Solving (6) for $x+y$, $x+y=9$.........(7). Substituting (7) in (3), (2), (1), $r=18$, $y=5$, $x=4$. For first month, \$1.50, for the second, \$1.20, and for the third, \$1.00 must be paid for each horse.

Hence, A owed \$9.30; B, \$12.00; C, \$14.50.

Also solved by O. L. Callecot, A. H. Holmes, L. E. Newcomb, J. Scheffer, G. B. M. Zerr, and the Proposer.

257. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x+y=10$, (2) $3x=\log_{10}y$.

I. Solution by HENRY HEATON, Belfield, N. D.

We have $3x=\log_{10}(10-x)$, or $10-x=10^{3x}$. If we suppose $x=\frac{1}{3}$, we obtain $9\frac{2}{3}=10$. An error of $\frac{1}{3}$. If we suppose $x=\frac{1}{4}$, we get $9.25=4.38$. An error of 4.87. By position we obtain $x=.328$. Substituting this we get $9.672=9.639$. An error of $-.043$. Substituting .329 for x we get $9.671=9.704$. An error of

.033. By position the second time we have $x=.3285$. Substituting this, we obtain $9.6715=9.6716$. An error of .0001.

Hence $x=.3285-$, and $y=10.-3285=9.6715+$.

II. Solution by A. H. HOLMES, Brunswick, Maine, and by J. SCHEFFER, Hagerstown, Md.

$x+y=10$ ....(1), $3x=\log_{10}y$ ....(2).

$\therefore \log y=\log(10-x)$ and $\log_{10}y=M\log(10-x)$. $\therefore \log(10-x)=3x/M$.

$$\log(10-x)=\frac{1}{M}+\log\left(1-\frac{x}{10}\right)=\frac{1}{M}-\frac{x}{10}-\frac{x^2}{200}-\frac{x^3}{3000}-\text{ etc. }=\frac{3x}{M}.$$

The series converges very rapidly and the first two terms are sufficient for a correct value of x to the fourth decimal figure.

$$\text{Hence }\frac{1}{.434294}-\frac{x}{10}-\frac{x^2}{200}=\frac{3x}{.434294}.$$

$\therefore x=.3285-$, and from (1), $y=9.6714+$.

III. Solution by S. A. COREY, Hiteman, Iowa.

From (1), $y=10-x$, and substituting in (2), $\log_{10}(10-x)=3x$ ....(3), whence $\log_{10}[10(1-\frac{x}{10})]=3x$ ....(4), and $\log_{10}10+\log_{10}(1-\frac{x}{10})=3x$ ..... (5). $\log_{10}(1-\frac{x}{10})=3x-1$ ....(6). But as $\log_{10}(1-\frac{x}{10})<0$, we have $3x<1$. Let $3x=1-3v$. By substituting in (6), $\log_{10}(\frac{29}{30}+\frac{v}{10})=-3v$, or $\log_{10}[\frac{29}{30}(1+\frac{3v}{29})]=-3v$, $\log_{10}(1+\frac{3v}{29})=\log_{10}30-\log_{10}29-3v=a-3v$ ($a=.014,723,256,8$) .... (7). But $\log_{10}(1+\frac{3v}{29})>0$, hence $3v<a$. Next let $3v=a-3w$. By substituting in (7), $\log_{10}\left(\frac{29+a}{29}-\frac{3w}{29}\right)=3w$, $\log_{10}\left[\left(\frac{29+a}{29}\right)\left(1-\frac{3w}{29+a}\right)\right]=3w$. Whence $\log_{10}\left(1-\frac{3w}{29+a}\right)=3w+\log_{10}29-\log_{10}(29+a)=3w-.000,220,434,6$ ....(8). But $\log_{10}\left(1-\frac{3w}{29+a}\right)=\frac{-3mw}{29+a}$ nearly, where m=modulus of common system of logarithms, and therefore (8) becomes $\frac{3mw}{29+a}+3w=.000,220,434,6$, nearly.

Whence, $w=.000,072,394,6$, nearly; and $x=\frac{1}{3}-a/3+w=.328,497,975,7$, $y=10-x=9.671,502,024,3$.

Also solved by G. W. Greenwood, G. B. M. Zerr, and the Proposer.

258. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum the infinite series $\dfrac{n^2}{(4n^2-1)^2}$ beginning with $n=1$, n being always odd.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n^2}{(4n^2-1)^2}=\tfrac{1}{16}\left[\frac{1}{(2n-1)^2}+\frac{1}{(2n+1)^2}-\frac{1}{2n+1}+\frac{1}{2n-1}\right]$$

Let $u=1, 3, 5, 7, \ldots\ldots\ldots$ Then

$$\Sigma\frac{n^2}{(4n^2-1)^2}=\tfrac{1}{16}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots\ldots\right]+\tfrac{1}{16}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\right.$$

$$\left.+\frac{1}{9}-\ldots\ldots\ldots\right].$$

$$\therefore\ \Sigma\frac{n^2}{(4n^2-1)^2}=\tfrac{1}{16}\cdot\frac{\pi^2}{8}+\tfrac{1}{16}\cdot\frac{\pi}{4}=\frac{\pi}{64}[\tfrac{1}{2}\pi+1]$$ (See line 1, p. 41).

Also solved by G. W. Greenwood, and Henry Heaton.

CALCULUS.

133. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Evaluate $\displaystyle\int\frac{\sqrt{(1+y)}}{1+y^2}dy$.

Solution by HENRY HEATON, Belfield, N. D.

Let $1+y=z^2$; then $\displaystyle\int\frac{\sqrt{(1+y)}}{1+y^2}dy=\int\frac{2z^2dz}{z^4-2z^2+2}=\frac{1}{2a}\int\Big(\frac{z}{z^2-2az+\sqrt{2}}$

$\displaystyle-\frac{z}{z^2+2az+\sqrt{2}}\Big)dz$, where $a=\sqrt{\dfrac{\sqrt{2}+1}{2}}$.

$$\frac{1}{2a}\int\Big(\frac{z}{z^2-2az+\sqrt{2}}-\frac{z}{z^2+2az+\sqrt{2}}\Big)dz=\frac{1}{2a}\int\Big(\frac{(z-a)+a}{z^2-2az+\sqrt{2}}$$

$$\frac{(z+a)-a}{z^2-2az+\sqrt{2}}\Big)dz=\frac{1}{2a}\log\sqrt{\frac{z^2-2az+\sqrt{2}}{z^2+2az+\sqrt{2}}}+\frac{1}{2\sqrt{[\sqrt{2}-a^2]}}\Big(\tan^{-1}\frac{z-a}{\sqrt{[\sqrt{2}-a^2]}}$$

$$+\tan^{-1}\frac{z+a}{\sqrt{[\sqrt{2}-a^2]}}\Big)=\sqrt{\frac{\sqrt{2}-1}{2}}\log\sqrt{\frac{1+y-\sqrt{[2\sqrt{2}+2]}\sqrt{[1+y]}+\sqrt{2}}{1+y+\sqrt{[2\sqrt{2}+2]}\sqrt{[1+y]}+\sqrt{2}}}$$

$$+\sqrt{\frac{\sqrt{2}+1}{2}}\tan^{-1}\Big(\frac{\sqrt{[2\sqrt{2}-2]}\sqrt{[1+y]}}{\sqrt{2}-1-y}\Big).$$

The above solution is in the real factors of z^4-2z^2+2.

On page 140, No. 5, Vol. IX, Dr. Zerr gives a solution, using the imaginary factors of z^4-2z^2+2. His final seemingly imaginary result reduces to a real value by substituting $\sqrt{\frac{1+\sqrt{2}}{2}}+\sqrt{\frac{1-\sqrt{2}}{2}}$ for $\sqrt{[1+\sqrt{(-1)}]}$ and $\sqrt{\frac{1+\sqrt{2}}{2}}-\sqrt{\frac{1-\sqrt{2}}{2}}$ for $\sqrt{[1-\sqrt{(-1)}]}$, reducing and remembering that

$$\log[a+b\sqrt{(-1)}]=\sqrt{(-1)}\tan^{-1}\frac{b}{a}+\log\sqrt{(a^2+b^2)}.$$

216. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Find the limit of the sum of the series

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots\dots+\frac{n}{n^2+m^2},$$

when n and m are indefinitely increased. (Distinguish the several cases arising from the different *relative* values of m and n.)

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots\dots+\frac{n}{n^2+m^2}$$

$$=h\left[1+\frac{1}{1+h^2}+\frac{1}{1+(2h)^2}+\frac{1}{1+(3h)^2}+\dots\dots+\frac{1}{1+(mh)^2}\right]$$

where $h=\frac{1}{n}$. Hence the required limit is $\int_0^{mh}\frac{dx}{1+x^2}=\tan^{-1}(mh)$.

If m is small compared with n so that $mh=0$, the limit is $\tan^{-1}0=0$. If m is equal to n so that $mh=1$, the limit is $\tan^{-1}1=\frac{1}{4}\pi$. If m is large compared with n so that $mh=\infty$, the limit is $\tan^{-1}\infty=\frac{1}{2}\pi$.

Also solved by S. A. Corey, and Henry Heaton.

217. Proposed by S. A. COREY, Hiteman, Iowa.

In *The Analyst*, Vol. II, p. 120, 1875, Dr. G. W. Hill finds by the method of mechanical quadrature the value of $\int_0^{\frac{1}{2}\pi}\frac{xdx}{\sin x[1+.16\cos^2x]^{\frac{3}{2}}}$ to be 1.6576363.

Evaluate the definite integral by some other method and verify above result.

Solution by the PROPOSER.

Using the formula* $f(x)-f(0)=\frac{x}{m.2}\left\{[f'(x)+f'(0)]+2\left[f'\left(\frac{x}{m}\right)\right.\right.$

*MONTHLY, Vol. 13, p. 12.

$$+f'\left(\frac{2x}{m}\right)+f'\left(\frac{3x}{m}\right)+\ldots+f'\left(\frac{m-1}{m}x\right)\Big]\Big\}-\frac{B_1x^2}{m^2.2!}[f''(x)-f''(0)]$$

$$+\frac{B_2x^4}{m^4.4!}[f^{\text{iv}}(x)-f^{\text{iv}}(0)]-\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}[f^{(2n)}(x)-f^{(2n)}(0)]+\ldots$$

(B_1, B_2, ..., B_n being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc.) and taking $m=6$, we get

$\frac{1}{24}\pi f'(\frac{1}{2}\pi)=\pi^2/48=$	.205,616,758
$\frac{1}{24}\pi f'(0)=\dfrac{\pi}{24(1.16)^{\frac{3}{2}}}=$	.104,773,547
$\frac{1}{12}\pi f'(\frac{1}{12}\pi)=\dfrac{\pi^2}{144\sin 15°(1+.16\cos^2 15°)^{\frac{3}{2}}}=$	.214,932,009
$\frac{1}{12}\pi f'(\frac{1}{6}\pi)=$	.231,297,112
$\frac{1}{12}\pi f'(\frac{1}{4}\pi)=$	.259,082,377
$\frac{1}{12}\pi f'(\frac{1}{3}\pi)=$	.298,480,940
$\frac{1}{12}\pi f'(\frac{5}{12}\pi)=$	.349,155,189
Sum	=1.663,337,932
$\dfrac{B_1(\frac{1}{2}\pi)^2}{6^2.2!}[f''(\frac{1}{2}\pi)-f''(0)]=\dfrac{\pi^2}{1728}=$	.005,711,577
	1.657,626,355
$\dfrac{B_2(\frac{1}{2}\pi)^4}{6^4.4!}[f^{\text{iv}}(\frac{1}{2}\pi)-f^{\text{iv}}(0)]=\dfrac{\pi^4}{30.12^4.4!}[3-9(.16)]=\dfrac{156\pi^4}{3000.12^4.4!}=$	.000,010,178
	1.657,636,533
$\dfrac{B_3(\frac{1}{2}\pi)^6}{6^6.6!}[f^{\text{vi}}(\frac{1}{2}\pi)-f^{\text{vi}}(0)]=\dfrac{\pi^6}{42.12^6.6!}[225(.16)^2-30(.16)+25]$	
$=\dfrac{2596\pi^6}{4200.12^6.6!}=$	.000,000,009
Value of integral correct to eight or 9 decimal places .	1.657,636,524
Value of integral obtained by Hill	1.657,636,3

As Hill used but seven decimal places in his computations we would not expect his result to be correct to the last decimal place. In the foregoing computations the term involving B_3 might have been omitted and the result would still have been more accurate than Hill's result, although his method involves somewhat more labor than does the foregoing.

219. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

In the article "Infinitesimal Calculus" in the *Encyclopaedia Britannica* Vol. XIII, page 24, I notice the following: "Of all triangular pyramids standing on a given triangular base, and of given altitude, find that whose surface is the least." A solution is required.

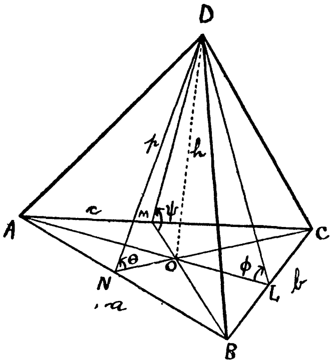
I. Solution by the PROPOSER.

Let ABC be the given triangular base, whose sides are a, b, c, and let h be the given altitude; also let θ, ϕ, ψ, be the angles of inclination of the faces of the pyramid to the base. Now, in the face DAB, if p is the perpendicular from the vertex on the side AB, then $\sin\theta=h/p$, hence $p=h\operatorname{cosec}\theta$.

But area of $\triangle DAB=\frac{1}{2}ap=\frac{1}{2}ah\operatorname{cosec}\theta$. Hence, the area of the *three* faces of the pyramid is

$$u=\tfrac{1}{2}h(a\operatorname{cosec}\theta+b\operatorname{cosec}\phi+c\operatorname{cosec}\psi)\ldots\ldots\ldots.(1).$$

Also, the base of the pyramid may be divided into three triangles, whose altitudes are easily found. For



since $\dfrac{h}{\text{altitude of }\triangle AOB}=\tan\theta$.

$\therefore$ Altitude of $\triangle AOB=h\cot\theta$.
$\therefore$ Area of $\triangle AOB=\frac{1}{2}ah\cot\theta$.
Hence the area of the *whole* base ABC is

$$v=\tfrac{1}{2}h(a\cot\theta+b\cot\phi+c\cot\psi)\ldots\ldots\ldots(2).$$

Now, equations (1) and (2) are perfectly general expressions for the surface and base of a pyramid with the given dimensions, and apply to any figure besides the one I have given. Now, from (1),

$$\frac{\partial u}{\partial\theta}=\frac{h}{2}\left[-a\operatorname{cosec}\theta\cot\theta-c\operatorname{cosec}\psi\cot\psi\frac{\partial\psi}{\partial\theta}\right]=0.$$

$$\frac{\partial u}{\partial\phi}=\frac{h}{2}\left[-b\operatorname{cosec}\phi\cot\phi-c\operatorname{cosec}\psi\cot\psi\frac{\partial\psi}{\partial\phi}\right]=0.$$

Whence, $a\operatorname{cosec}\theta\cot\theta\dfrac{\partial\psi}{\partial\phi}=-c\operatorname{cosec}\psi\cot\psi\dfrac{\partial\psi}{\partial\theta}\cdot\dfrac{\partial\psi}{\partial\phi}$,

and $b\operatorname{cosec}\phi\cot\phi\dfrac{\partial\psi}{\partial\theta}=-c\operatorname{cosec}\psi\cot\psi\dfrac{\partial\psi}{\partial\phi}\cdot\dfrac{\partial\psi}{\partial\theta}$.

$$\therefore a\operatorname{cosec}\theta\cot\theta\frac{\partial\psi}{\partial\phi}=b\operatorname{cosec}\phi\cot\phi\frac{\partial\psi}{\partial\theta}\ldots\ldots\ldots(3).$$

From (2), $\dfrac{2v}{h}=a\cot\theta+b\cot\phi+c\cot\psi$. $\therefore c\cot\psi=\dfrac{2v}{h}-a\cot\theta-b\cot\phi$.

Differentiating successively as to θ and ϕ, we have

$$-c\,\text{cosec}^2\,\psi\,\frac{\partial\psi}{\partial\theta}=a\,\text{cosec}^2\,\theta,\ \text{and}\ -c\,\text{cosec}^2\,\psi\frac{\partial\psi}{\partial\phi}=b\,\text{cosec}^2\,\phi.$$

Whence, $\dfrac{\partial\psi}{\partial\theta}=-\dfrac{a\,\text{cosec}^2\,\theta}{c\,\text{cosec}^2\,\psi},\quad \dfrac{\partial\psi}{\partial\phi}=-\dfrac{b\,\text{cosec}^2\,\phi}{c\,\text{cosec}^2\,\psi}.$

Substituting these values in (3), we get

$$a\,\text{cosec}\,\theta\cot\theta\,\frac{b\,\text{cosec}^2\,\phi}{c\,\text{cosec}^2\,\psi}=b\,\text{cosec}\,\phi\cot\phi\,\frac{a\,\text{cosec}^2\,\theta}{a\,\text{cosec}^2\,\psi}.$$

$\therefore\cot\theta\,\text{cosec}\,\phi=\cot\phi\,\text{cosec}\,\theta.\quad \cos\theta=\cos\phi;\ \therefore\theta=\phi.$

Similarly, by finding $\dfrac{\partial u}{\partial\theta}, \dfrac{\partial u}{\partial\psi}$, it may be shown that $\theta=\psi$. Hence, $\theta=\phi=\psi$, and the faces of the pyramid must be equally inclined to the base.

As it is evident from the nature of the problem that there is a minimum, it is unnecessary to proceed to the higher derivatives.

The results can be generalized to apply to a pyramid of any number of faces, but the work is a little more complicated.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC represent the given base, and $DO=h$ the given altitude. In the base ABC, from O let fall the three perpendiculars x, y, z upon the sides BC, AC, and AB, respectively. Then the lateral surface will be$=\frac{1}{2}a\sqrt{(h^2+x^2)}+\frac{1}{2}b\sqrt{(h^2+y^2)}+\frac{1}{2}c\sqrt{(h^2+z^2)}$.

Consequently, $M=a\sqrt{(h^2+x^2)}+b\sqrt{(h^2+y^2)}+c\sqrt{(h^2+z^2)}$ is to be a minimum, subject to the relation $ax+by+cz=2\triangle$, $\triangle$ denoting the area of $\triangle ABC$. From the first of these equations we get, by differentiation

$$\frac{\partial M}{\partial x}=\frac{ax}{\sqrt{(h^2+x^2)}}+\frac{cz}{\sqrt{(h^2+y^2)}}\frac{\partial z}{\partial x},\quad \frac{\partial M}{\partial y}=\frac{by}{\sqrt{(h^2+y^2)}}+\frac{cz}{\sqrt{(h^2+z^2)}}\frac{\partial z}{\partial y},$$

and from the second we get

$$\frac{\partial z}{\partial x}=-\frac{a}{c},\quad \frac{\partial z}{\partial y}=-\frac{b}{c}.$$

Substituting, and putting $\dfrac{\partial M}{\partial x}, \dfrac{\partial M}{\partial y}$ each$=0$, we obtain

$$\frac{x}{\sqrt{(h^2+x^2)}}=\frac{z}{\sqrt{(h^2+z^2)}}=\frac{y}{\sqrt{(h^2+y^2)}}.$$

Hence $x=y=z$. The pyramid of a minimum surface is therefore the one whose faces are equally inclined to the base.

DIOPHANTINE ANALYSIS.

133. Proposed by REV. R. D. CARMICHAEL, Hartselle, Alabama.

Find all perfect numbers of four primes and of multiplicity 4.

Solution by the PROPOSER.

The object of this note is to show that $2^5.3^3.5.7=30240$ is the only multiply perfect number* of multiplicity 4 and having only 4 distinct primes.

Let $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}p_4^{a_4}$ be the number where the primes p_1, p_2, p_3, p_4, are in order of magnitude, beginning with the smallest. The sum of the divisors is equal to four times the number. Dividing this equation by the number, have

$$4=\frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)}\cdot\frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)}\cdot\frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}\cdot\frac{p_4^{a_4+1}-1}{p_4^{a_4}(p_4-1)}\text{ ....}(1).$$

$\therefore 4<\dfrac{p_1}{p_1-1}\cdot\dfrac{p_2}{p_2-1}\cdot\dfrac{p_3}{p_3-1}\cdot\dfrac{p_4}{p_4-1}$ .... (2), from which it may easily be shown that $p_1=2$, $p_2=3$, $p_3=5$, and $p_4=7$, 11, or 13. From (1) we may write (for use later)

$$4<\frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{p_4}{p_4-1}\text{ .... }(3).$$

Again from (1),

$$4=\frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{3^{a_2+1}-1}{3^{a_2}.2}\cdot\frac{5^{a_3+1}-1}{5^{a_3}.4}\cdot\frac{p_4^{a_4+1}-1}{p_4^{a_4}(p_4-1)}\text{ ....}(4).$$

$$\therefore 2^{a_1+5}.3^{a_2}.5^{a_3}.p_4^{a_4}(p_4-1)=(2^{a_1+1}-1)(3^{a_2+1}-1)(5^{a_3+1}-1)(p_4^{a_4+1}-1)\text{ ..... }(5).$$

Obviously n may be so taken that

$$2^{a_1+1}-1=(2^{2n+1}-1)(2^{2n+1}+1)(2^{2(2n+1)}+1)(2^{4(2n+1)}+1)\text{ ...., ... }(6)$$

where limitations are to be found for the series of factors. The fourth factor introduces the prime 17, and hence not more than 3 factors may be used. We may easily show that $2^{2n+1}-1$ is not divisible by 3 or 5; and also that it is not a power of $p_4=11$ or 13; and that if it is a power of $p_4=7$, it is the first power. Hence, if $p_4=11$ or 13, $n=0$, and we have from equation (6), $a_1=1$ or 3. These values of p_4 and a_1 will not satisfy equation (3), however they are combined. Now, if $p_4=7$, $n=0$ or 1 (by the preceding). The possible values of a_1 are found from (6) to be $a_1=1$, 2, 3, 5, 7. When $a_1=7$ the prime 127 is introduced, and this value must therefore be discarded. We therefore have left to consider the cases of $p_4=7$ and $a_1=1$, 2, 3, or 5. Substituting $p_4=7$ in equation (5) we have

*This term was introduced by D. N. Lehmer in 1901; see *Annals of Mathematics*, Ser. 2, Vol. 2, p. 103. "A multiply perfect number is one which is an exact divisor of the sum of all the divisors, the quotient being the multiplicity."

$$2^{a_1+6}.3^{a_2+1}.5^{a_3}.7^{a_4}=(2^{a_1+1}-1)(3^{a_2+1}-1)(5^{a_3+1}-1)(7^{a_4+1}-1)\ldots\ldots(7).$$

Now n may be so taken that

$$3^{a_2+1}-1=(3^{2n+1}-1)(3^{2n+1}+1)(3^{2(2n+1)}+1)(3^{4(2n+1)}+1)\ldots\ldots,\ \ldots\ldots(8).$$

The last factor written introduces the prime 41, and therefore not more than three factors can be considered. It may easily be shown that $3^{2n+1}-1$ is not divisible by 5 or 7, and that it contains the factor 2 but once. Therefore, $3^{2n+1}-1=2$. Hence, $n=0$. Equation (8) now yields $a_2=1$ or 3.

A trial of each of the eight possible cases produced by every possible combination of the values $a_2=1$ or 3, and $a_1=1$, 2, 3, or 5, will result in finding but one number of the type here considered, namely, $2^5.3^3.5.7$.

MECHANICS.

187. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

Solution by G. W. GREENWOOD, M. A., McKendree College. Lebanon, Ill.

Take coördinate axes through the center of force, and let μ^2 be the force on a particle of unit mass at the unit distance. Then the equations of motion are

$$\frac{d^2x}{dt^2}=-\mu^2 r\cos\theta=-\mu^2 x,\quad \frac{d^2y}{dt^2}=-\mu^2 r\sin\theta=-\mu^2 y.$$

Integrating the equations of motion we have

$$x=a\cos\mu t+b\sin\mu t,\quad y=c\cos\mu t+d\sin\mu t.$$

$\therefore (cx-ay)^2+(dx-by)^2=(bc-ad)^2$, which is an ellipse.

Also solved by S. A. Corey, and Henry Heaton.

PROBLEMS FOR SOLUTION.

ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\frac{n}{(4n^2-1)^2}$, beginning with $n=1$, n being always odd.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals e and π in the form of infinite continued fractions.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant $2(a_0a_4-4a_1a_3+3a_2^2)$ of the binary quartic $a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2+4a_3x_1x_2^3+a_4x_2^4$ in terms of roots of the latter.

AVERAGE AND PROBABILITY.

177. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random planes cut a given sphere. What is the chance that they intersect within the sphere?

CALCULUS.

220. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

To determine the least polygon of n sides that can be described about a given circle.

DIOPHANTINE ANALYSIS.

135 Proposed by A. H. HOLMES, Brunswick, Maine.

In the equation in Diophantine Analysis: $2x^2+2x+1=\square=u^2$, show that u is always the sum of two squares.

136. Proposed by A. H. HOLMES, Brunswick, Maine.

Given $7x^2-111=y^2$. Required a value for y greater than unity which shall be a prime integer.

GEOMETRY.

288. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

From a point P on a given circle to draw two chords such that, (α) chord PA : chord $PB=m$: n (a given ratio), and, (β) arc PA : arc $PB=1$: 3.

289. Proposed by J. J. QUINN, Ph. D., Warren, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

290. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Show that the point (1, 1) is a conjugate point on the locus $x^3+y^3-3xy+1=0$.

MISCELLANEOUS.

158. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

In ingot of pure gold was melted at the Mint and then 10 ounces were taken out and 10 ounces of pure silver added and the contents of the melting pot mixed thoroughly. This was repeated until there were 10 such operations in all. The contents of the pot being then assayed was found to be nine-tenths fine, or standard gold. What was the weight of the original ingot? There was no loss in the precious metals by the melting.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Diophantine Analysis, No. 132. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

From the numbers, 0, 1, 2, ..., 42, select seven, such that the 42 differences of these seven numbers shall be congruent (mod. 43) to the numbers 0, 1, 2, ..., 42. The differences may be both positive and negative.*

Mechanics, No. 188. Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in Educational Times, London.)

Spherical bubbles are rising in water. Find the relation between radius and velocity.

NOTES AND NEWS.

Professor J. H. Jeans of Princeton University, has been elected fellow of the Royal Society of London.

Dr. E. B. Wilson has been promoted to an assistant professorship of mathematics at Yale, University.

Dr. Oliver E. Glenn has been appointed instructor in mathematics in the University of Pennsylvania.

Professor W. J. Hussey, of Lick Observatory, has been appointed professor of astronomy at the University of Michigan.

*These problems involve important principles. Solutions have been contributed, but all incorrect. Will some reader make a study of them? ED. G.

The death is announced of Professor James Mills Peirce, Perkins Professor of Mathematics and Astronomy at Harvard University.

Mr. Louis A. Martin, Jr., has been promoted from instructor to assistant professor of mathematics and mechanics in Stevens Institute of Technology.

The April number of *Annals of Mathematics* contains "Theory of Fourier's Series" (concluded), by M. Bôcher; "Note on Multiply Perfect Numbers," by R. D. Carmichael; "Note on Integrating Factors," by Edwin Bidwell Wilson.

The latter part of Dr. Halsted's essay "The Value of Non-Euclidean Geometry" (*Popular Science Monthly*, November, 1905) has been reprinted in New Zealand with added notes by F. W. Frankland [Okataina, Foxton, Manawatu, New Zealand].

The American Association for the Advancement of Science will hold a summer session at Ithaca, New York, during the week June 26 to July 2. Dr. Edward Kasner is vice president, and Professor L. G. Weld secretary of Section A, Mathematics and Astronomy.

Dr. S. T. Tamura, mathematician in the department of terrestrial magnetism of the Carnegie Institution, has been offered a professorship of dynamics and ship's magnetism in the Naval Staff College, Tokyo, which is the graduate school for Japanese naval officers.

The one hundred and twenty-eighth regular meeting of the American Mathematical Society was held at Columbia University, New York City, on April 28th. The nineteenth meeting of the Chicago Section of the Society convened at Northwestern University, on April 14th. A total of thirty-nine papers were presented at these meetings.

The Thirteenth Summer Meeting and Fifth Colloquium of the American Mathematical Society will be held at Yale University during the entire week of September 3—8, 1906. The Colloquium which will open on Wednesday morning, will include the following courses of lectures: Professor E. H. Moore, "On the theory of bilinear functional operators;" Professor Max Mason, "Selected topics in the theory of boundary value problems of differential equations;" Professor E. J. Wilczynski, "Projective differential geometry."

JAMES MILLS PEIRCE. The Faculty of Arts and Sciences of Harvard University have adopted the following minutes on the life and services of the late Professor Peirce:

"The Faculty of Arts and Sciences desire to put on record their sense of great loss which they have sustained in the death of Professor James Mills Peirce.

"Born in Cambridge within sound of the College bell, a member of the faculty of Harvard College at twenty, serving for nearly fifty years, not as teacher merely, but successively as secretary of the academic council, as dean—

and almost as father—of the graduate school, and as dean of the faculty of arts and sciences, he spent the whole of a long life in and for the University.

"He was an admirable teacher, steeped in his subject, not buried in it, and always in close sympathy with his students, to whom he was ever a generous and inspiring friend. Broad minded and many sided, his scholarship was of that wide, human kind which unites learning with recognition of every accomplishment of grace of life, with interest in every intellectual problem, and with good will to every earnest man. All his work was characterized by thoroughness and finish, and by a kind of fervid loyalty. He had a high and large conception of academic freedom, and, in age as in youth, he looked forward and not back. Of a peculiarly lovable nature, courteous and kindly, he was known to all who met him for his friendly greeting, his earnest speech, at once measured and impetuous, and his scorn of anything narrow, or crooked, or mean."

At the University of Wisconsin, Dr. Edward B. VanVleck, now professor of mathematics at Wesleyan University, has been appointed to the professorship of mathematics left vacant by the resignation of Professor E. A. VanVelzer.

BOOK NOTICES.

Elementary Algebra. By G. A. Wentworth. Half Leather, vi+421 pages. Boston, New York, Chicago: Ginn & Co.

One of the best features of this book is a new set of exercises, some four thousand in number, with which the author has supplied the text. The appearance of the volume is attractive; the use of colors in plotting equations may be commended. The author shows a predilection for using more or less useless technical terms, "compound expressions" for polynomials, "scalar numbers" for real numbers, "orthotomics" for imaginaries. The real and imaginary axes are said to intersect "orthotomically." This thing should be avoided in an elementary algebra. The book would be more readable by student and teacher if divested of some inaccuracies. We read on page 1, "Whatever admits of increase or decrease is called *a magnitude.*" The length of a meter stick for example! On page 289 the student learns that "Gravity always impresses upon a body, free to move, a downward velocity of g units each second, whether the body starts from a state of rest or is moving already with any velocity in any direction."

The treatment of Factoring is adequate, as is the graphical work in connection with simultaneous equations and imaginaries. But eleven pages are devoted to solving quadratics by completing the square, while the quadratic formula gets barely three pages, part fine print. The idea of a function is rather difficult for secondary students, but its introduction in the author's excellent section on variation may find justification. The section on "Laws of Physics" should not precede the treatment of variation. On the whole *Elementary Algebra* is not a contribution to mathematical pedagogy, but it is a teachable book and no doubt will meet with considerable success.

An Elementary Text-Book of Theoretical Mechanics. By George A. Merrill, B. S., Principal of the California School of Mechanical Arts, and Director of the Wilmerding School of Industrial Arts. Half Leather, 268 pages, 168 diagrams. San Francisco, New York, Cincinnati, Chicago: American Book Co.

This book is intended for upper classes in secondary schools and lower classes in colleges. It is almost alone in its field, so far as American publications are concerned. The teaching of Mechanics as a subject *per se* has been confined in the main to collegiate courses, and the few American text-books on the subject have been written for students familiar with the calculus. There is need of giving greater prominence to this subject in secondary schools, especially in institutions whose graduates look forward to industrial careers. Professor Merrill's book is most admirably suited to supply this need. The text is divided into three principal sections, Kinematics, Statics, and Kinetics. It is written "from the standpoint of the student," is clear, forceful, and pedagogical. Vectors are introduced at the start, and graphical methods and illustrations are used profusely and effectively, throughout the book. The attractiveness of the work is much enhanced by the use of several kinds of type, and by the large number of excellent illustrations. Good full page portraits of Galileo, Hugenius, and Newton adorn the work.

A Brief Introduction to the Infinitesimal Calculus. By Irving Fisher, Ph. D., Professor of Political Economy in Yale University. Second Edition. Cloth, 84 pages. London: Macmillan and Co.

The author states in his preface that this little book contains the substance of lectures by which he has been accustomed to introduce advanced students to a course in modern economic theory. Theprocesses of differentiation and integration are introduced and illustrated by elementary applications.

A First Course in Physics. By Robert Andrews Millikan, Ph. D., Assistant Professor of Physics in the University of Chicago, and Henry Gordon Gale, Ph. D., Instructor of Physics in the University of Chicago. Boston, New York, Chicago, London: Ginn & Co.

The advance pages of this new secondary school physics indicate that the authors are using the best efforts of scholarship to produce a text that will answer the demand of the day that the first course in physics be not only completely scientific, but be full of life and interest for the student as well. Judging from the first hundred pages of the book, clear and simple English, profuseness of illustration, and constant reference and application to actual natural phenomena are some of the strong features which are to characterize *A First Course in Physics.*

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. MAY, 1906. No. 5.

ON THE n-SECTION OF AN ANGLE.

By REV. R. D. CARMICHAEL, Hartselle, Alabama.

The locus of the polar equation $r=a\cos\dfrac{n-1}{n}\theta$ is readily applicable to the problem of the n-section of angle. We plot a segment of the curve for $n=3$.

Let $OA=a$, and let O be the origin. Construct the circular segment AQM with O as center. The given locus and the circle coincide at A. Place the angle whose nth part is required in the position OAP, P a point on the given locus. Draw PQ perpendicular to OP. The angle AOQ is the required nth part of AOP. For, $OP=a\cos\dfrac{n-1}{n}\theta$, θ the angle AOP; hence $PQ=a\sin\dfrac{n-1}{n}\theta$; and therefore $\angle POQ=\dfrac{n-1}{n}\theta$; then, $\angle AOQ=\dfrac{\theta}{n}$. For the case constructed, $n=3$, this gives the trisection of the angle.

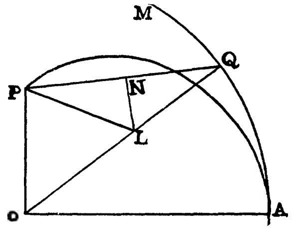


Fig. 1.

It is desirable to have some method of describing this curve by continuous motion. Let L in the above figure be the middle point of OQ. Then, since OPQ is a right angle, $PL=OL=LQ$. Draw LN perpendicular to PQ. It is parallel to OP. Therefore $\angle NLQ=\angle POQ=\frac{1}{2}\angle PLQ$. Hence, $\angle PLQ=\dfrac{2(n-1)}{n}\theta$. Now, let a material circle be fixed to PL with center at L. Also, let another circle

whose radius is $2(n-1)$ times that of the former be fixed to the plane of the paper with center at O. Let a tight band pass around both circles crossing between them. It is evident now that, as OQ revolves about O, the point P will describe the locus in question. This follows from the fact that always $\angle AOQ=\frac{\theta}{n}$, and $\angle PLQ=\frac{2(n-1)}{n}\theta$.

Now, if the fixed circle at O is of radius $n-1$, and the circle fixed at L is of radius $\frac{1}{2}$, the locus of P is another curve whose equation is $r=a\cos\frac{\theta}{n}$; and by means of this curve we may in an identical manner effect the n-section of the angle, in this case cutting off the smaller section from the other side of the angle.

Another curve for the same purpose may be described as follows:

Let OA and AB be two rods, each of length a, jointed at A, and let OA be fixed at O. Let a circle of radius n be fixed to the plane of the paper at O, and let another circle of radius $n-1$ be fixed to AB with center at A. Pass a tight band around the two circles *not* crossing between them. As AO revolves about O, AB revolves about A in the opposite direction. If $\angle POA=\theta$ $(=\angle QAM)$, then $\angle BAM=\frac{n-1}{n}\theta$.

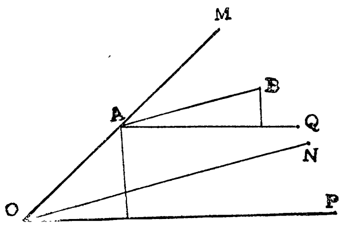


Fig. 2.

Therefore, $\angle BAQ=\frac{\theta}{n}$. At O draw OC parallel to AB. $\angle POC=\frac{1}{n}\cdot\angle POA$, and the n-section of an angle is therefore readily effected. For, place the given angle in the position POM. Lay off $OA=a$; with a radius a and A as center, draw an arc cutting the given locus at B. Draw AB, and also OC parallel to AB.

The equations of this curve are readily expressed by means of the auxiliary angle θ. They are

$$x=a(\cos\theta+\cos\frac{\theta}{n}),\quad y=a(\sin\theta+\sin\frac{\theta}{n}).$$

If the circle attached at A has a radius 1 instead of $n-1$, the equations of the corresponding locus are

$$x=a(\cos\theta+\cos\frac{n-1}{n}\theta),\quad y=a(\sin\theta+\sin\frac{n-1}{n}\theta).$$

By means of this locus also the n-section of an angle is readily effected in a manner similar to the last method above.

THE APPROXIMATE SUMMATION OF n TERMS OF ANY HARMONIC SERIES.

By O. L. CALLECOT, Omaha, Nebraska.

For any series $U_1+U_2+\text{.........}$, we shall give notations to certain groups of successive terms; thus,

$$G_1=U_{m+1}+U_{m+2}+\text{.........}+U_{m+x_1},\ G_2=U_{m+x_1+1}+\text{.........}+U_{m+x_1+x_2},\ \text{.........}$$

Then for $n>m$, we have

$$(1)\qquad E\equiv(G_1+\text{.........}+G_n)-(U_1+\text{.........}+U_n)=(G_1-U_1)+\text{.........}+(G_n-U_n),$$

$$(2)\qquad E=(U_{n+1}+U_{n+2}+\text{.........}+U_{m+x_1+\ldots+x_n})-(U_1+U_2+\text{.........}+U_m),$$

since the terms U_{m+1}, ..., U_n are common to $G_1+\text{.........}+G_n$ and $U_1+\text{.........}+U_n$.

Applying this method to the special harmonic series $1+\frac{1}{2}+\frac{1}{3}+\text{.........}$, and taking $m=1$, $x=y=w=\text{.........}=z=3$, we have, by (1) and (2),

$$(3)\qquad E=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\text{.....}+\left(\frac{1}{3n-1}+\frac{1}{3n}+\frac{1}{3n+1}\right).$$

$$-\left(1+\frac{1}{2}+\text{.....}+\frac{1}{n}\right)=\frac{2}{2.3.4}+\frac{2}{5.6.7}+\text{.....}+\frac{2}{(3n-1)(3n)(3n+1)},$$

$$(4)\qquad E=\frac{1}{n+1}+\frac{1}{n+2}+\text{.....}+\frac{1}{3n+1}-1.$$

$$(5)\qquad \text{Hence}\ \frac{1}{n+1}+\text{.....}+\frac{1}{3n+1}=1+\frac{2}{2.3.4}+\text{.....}+\frac{2}{(3n-1)(3n)(3n+1)}.$$

But

$$(6)\qquad \frac{2}{2.3.4}+\frac{2}{5.6.7}+\text{.....}\infty=.098612\text{.........}$$

$$(7)\qquad \frac{2}{2.3.4}+\text{.....}+\frac{2}{(3n-1)(3n)(3n+1)}=.098612\text{.....}-\frac{1}{3(3n+1)(3n+2)}+\delta,$$

$$\delta<\frac{2}{(3n+1)(3n+2)(3n+3)(3n+4)}.$$

Now any number which gives the remainder 1 on division by 3 may be taken as the last in a group of $2n+1$ terms. We may therefore arrange the series $1+\frac{1}{2}+\text{.........}$ in successive groups of $2n+1$ terms by throwing out 0, 1, or 2 terms for each group, as the case may be.

(8) Hence $1+\frac{1}{2}+\dots+\frac{1}{n'}=G+D+U$,

where G is the number of groups in n' terms, U is the sum of the deleted terms, and D is the sum of the difference groups, $\frac{2}{2.3.4}+\dots+\frac{2}{(3n-1)(3n)(3n+1)}$. If n is not taken smaller than 15,

(9) $$D=(.098612\dots)G-\frac{10^G-1}{27(10^{G-1})(3g+1)(3g+2)}-\delta,$$

$$\delta<\frac{1}{216(3g+1)(3g+2)},$$

where g is the n of the group of lowest denominator.

Since the formula applies to the series $\frac{1}{r}+\frac{1}{r+1}+\dots$, $r>15$, we apply the following table for the sum of the earlier terms.

TABLE GIVING SUM OF FIRST n TERMS OF SERIES $1+\frac{1}{2}+\dots$

n	*Sum*	n	*Sum*
2	1.5	23	3.734291
3	1.833333	24	3.775958
4	2.083333	25	3.815958
5	2.283333	26	3.854419
6	2.449999	27	3.891456
7	2.592857	28	3.927171
8	2.717857	29	3.961653
9	2.828968	30	3.994987
10	2.928968	31	4.027245
11	3.019877	32	4.058495
12	3.103210	33	4.088798
13	3.180133	34	4.110209
14	3.251562	35	4.138781
15	3.318228	36	4.166559
16	3.380728	37	4.193586
17	3.439552	38	4.219901
18	3.495108	39	4.245542
19	3.547739	40	4.270542
20	3.597739	41	4.294933
21	3.645338	42	4.318742
22	3.690813	43	4.341998

Example. Find the sum approximately of $1+\frac{1}{2}+\ldots\ldots+\frac{1}{10,000}$.

	Denominators of terms in U.	
10000		
3) 9999	0	0
3333	3333	3332
3331		
3) 3330		
1110	1110	1109
1108		
3) 1107		
369	369	368
367		
3) 366		
122	122	0
121		
3) 120; ——40		

$B=4.270542$

$U=0.016088$

$(1.098612)5=5.493061$

9.779692

Second term of (9) $=-0.000025$

Sum $=9.77966 \ldots\ldots\ldots -\delta,\ \delta<.000,000,313$.

Any harmonic series $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots\ldots$ may be reduced to the form

$$\frac{1}{d}\left(\frac{1}{n+f}+\frac{1}{n+1+f}+\frac{1}{n+2+f}+\ldots\ldots\ldots\right)$$

where f is the fractional part of the result of dividing a by d, and n is the integral part of the result. Also,

$$\frac{1}{d}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots\ldots+\frac{1}{n'}\right)-\frac{1}{d}\left(\frac{1}{n+f}+\frac{1}{n+1+f}+\ldots\ldots+\frac{1}{n'+f}\right)$$

$$=\frac{1}{d}\left(\frac{f}{n(n+f)}+\frac{f}{(n+1)(n+1+f)}+\ldots\ldots+\frac{f}{n'(n'+f)}\right).$$

Since the value of $\frac{1}{d}\left(\frac{1}{n}+\ldots\ldots+\frac{1}{n'}\right)$ may be found according to methods already

given, and since the series $\frac{1}{d}\left(\frac{f}{n(n+f)}+\text{.........}\right)$ converges rapidly, we may approximately determine the sum of n terms of any harmonic series.

NOTE ON THE MAXIMUM INDICATOR OF CERTAIN ODD NUMBERS.

By REV. R. D. CARMICHAEL, Hartselle, Alabama.

If p, the least prime factor of N, is of the form $4l+1$, the maximum indicator of N is a multiple of 4.

In the MONTHLY, May, 1905, p. 107, I have shown that, for p a prime of the form $4l+1$, we have

$$\left(1.2.3\text{.........}\frac{p-1}{2}\right)^2\equiv -1 \pmod p.$$

Hence, if $\left(1.2.3\text{.........}\frac{p-1}{2}\right)^{2n}\equiv 1 \pmod N$, n is even; and therefore the maximum indicator of N, being a multiple of the least value $2n$ satisfying the above congruence, is also a multiple of 4.

Corollary. *The equation $y^4=mx+1$ has at least one positive integral solution when the least factor of m is congruent to unity modulo 4.*

By Wilson's theorem, it is easily shown as above that

The maximum indicator of any odd number is even.

Corollary. *The equation $y^2=mx+1$ has at least one positive integral solution, when m is odd.*

If p and $2p-1$ are odd primes, the maximum indicator of $p(2p-1)$ is a multiple of 4.

As in the first congruence above we have

$$(1.2.3\text{.........}\overline{p-1})^2\equiv -1 \pmod{2p-1}.$$

By Wilson's theorem,

$$(1.2.3\text{.........}\overline{p-1})^2\equiv 1 \pmod p.$$

$$\text{Thus } (1.2.3\text{.........}\overline{p-1})^4\equiv 1 \pmod{p.\overline{2p-1}}.$$

Now, $(1.2.3\text{.........}\overline{p-1})$ is not congruent to 1 modulo $p.\overline{2p-1}$. It is thus shown that one indicator, at least, is 4. Hence the maximum indicator is a multiple of 4.

ON A FUNDAMENTAL THEOREM IN TRIGONOMETRY.

By G. A. MILLER.

Many of our elementary text-books on trigonometry give the values of the trigonometric functions of the following eight angles: $\pm x$, $90 \pm x$, $180 \pm x$, $270 \pm x$. It has been observed that these eight angles may be derived from any one of them by means of the operations of a well known group which is known as the octic group, or the group of the square.* The present note is intended to exhibit more clearly the value of this point of view by presenting the matter in a somewhat different form from the one given in the article to which reference has been made.

We shall represent by c the operations of taking the complement, while p will be used to represent that the angle is increased by 90°. As angles which differ as a multiple of 360° are considered identical these operations are of periods 2 and 4, respectively. In fact the first is of period 2 independently of the modulus. The operations of increasing an angle by 180° and by 270° will be denoted bp p^2 and p^3, respectively, since these results are obtained by increasing the angle twice or three times by 90°. In accord with this $p^5=p$, $c^3=c$, $p^4=c^2=1$.

The eight operations of the octic group may be written as follows:

$$\begin{array}{cccc} 1 & p & p^2 & p^3 \\ c & cp & cp^2 & cp^3 \end{array}$$

If these operations are performed on the angle x the resulting angles are respectively:

$$\begin{array}{cccc} x & x+90^\circ & x+180^\circ & x+270^\circ \\ 90^\circ-x & 180^\circ-x & 270^\circ-x & -x \end{array}$$

By means of these results the functions of the eight angles mentioned above may be expressed in terms of functions of x provided we know the functions of $cx=90-x$, and of $px=x+90^\circ$. The former functions are simply a matter of definition while the latter are given in the following table:

$$\begin{aligned} \sin px &= \sin(90^\circ+x) = \cos x \\ \cos px &= \cos(90^\circ+x) = -\sin x \\ \tan px &= \tan(90^\circ+x) = -\cot x \\ \cot px &= \cot(90^\circ+x) = -\tan x \\ \sec px &= \sec(90^\circ+x) = -\csc x \\ \csc px &= \csc(90^\circ+x) = \sec x. \end{aligned}$$

*Quarterly Journal of Mathematics, Vol. 37 (1906), p. 226.

Hence,

$$
\begin{aligned}
&\sin p^2x=\sin\,(180^\circ+x)=\cos px=-\sin x^{*}\\
&\cos p^2x=\cos\,(180^\circ+x)=-\sin px=-\cos x\\
&\tan p^2x=\tan\,(180^\circ+x)=-\cot px=\tan x\\
&\cot p^2x=\cot\,(180^\circ+x)=-\tan px=\cot x\\
&\sec p^2x=\sec\,(180^\circ+x)=-\csc px=-\sec x\\
&\csc p^2x=\csc\,(180^\circ+x)=\sec px=-\csc x.
\end{aligned}
$$

$$
\begin{aligned}
&\sin p^3x=\sin\,(270^\circ+x)=\cos p^2x=-\cos x^{\dagger}\\
&\cos p^3x=\cos\,(270^\circ+x)=-\sin p^2x=\sin x\\
&\tan p^3x=\tan\,(270^\circ+x)=-\cot p^2x=-\cot x\\
&\cot p^3x=\cot\,(270^\circ+x)=-\tan p^2x=-\tan x\\
&\sec p^3x=\sec\,(270^\circ+x)=-\csc p^2x=\csc x\\
&\csc p^3x=\csc\,(270^\circ+x)=\sec p^2x=-\sec x.
\end{aligned}
$$

$$
\begin{aligned}
&\sin cpx=\sin\,(180^\circ-x)=\cos cx=\sin x\\
&\cos cpx=\cos\,(180^\circ-x)=-\sin cx=-\cos x\\
&\tan cpx=\tan\,(180^\circ-x)=-\cot cx=-\tan x\\
&\cot cpx=\cot\,(180^\circ-x)=-\tan cx=-\cot x\\
&\sec cpx=\sec\,(180^\circ-x)=-\csc cx=-\sec x\\
&\csc cpx=\csc\,(180^\circ-x)=\sec cx=\csc x.
\end{aligned}
$$

$$
\begin{aligned}
&\sin cp^2x=\sin\,(270^\circ-x)=-\sin cx=-\cos x^{\ddagger}\\
&\cos cp^2x=\cos\,(270^\circ-x)=-\cos cx=-\sin x\\
&\tan cp^2x=\tan\,(270^\circ-x)=\tan cx=\cot x\\
&\cot cp^2x=\cot\,(270^\circ-x)=\cot cx=\tan x\\
&\sec cp^2x=\sec\,(270^\circ-x)=-\sec cx=-\csc x\\
&\csc cp^2x=\csc\,(270^\circ-x)=-\csc cx=-\sec x.
\end{aligned}
$$

$$
\begin{aligned}
&\sin cp^3x=\sin -x=-\cos cx=-\sin x\\
&\cos cp^3x=\cos -x=\sin cx=\cos x\\
&\tan cp^3x=\tan -x=-\cot cx=-\tan x\\
&\cot cp^3x=\cot -x=-\tan cx=-\cot x\\
&\sec cp^3x=\sec -x=\csc cx=\sec x\\
&\csc cp^3x=\csc -x=-\sec cx=-\csc x
\end{aligned}
$$

The advantages of the present view point result from the fact that it associates the group and these eight angles in such a way that a knowledge of the one throws light on the other. If we know the octic group we know directly what two angles may be made fundamental in the study of the others. The necessary and sufficient condition is that the two angles correspond to generators in

*Sin $p^2x=\sin pz=\cos z$, where $z=px$.

†Sin $p^3x=\sin py=\cos y$, where $y=p^2x$; also, sin $p^3x=\sin p^2w=-\sin w$, where $w=px$

‡Also, sin $cp^2x=\sin py=\cos y$, where $y=cpx$, etc.

the octic group. In other words they must correspond to two non-commutative operators of this group since any two non-commutative operators of a non-abelian group of order p^3 generate the group and any two generators are non-commutative.

The properties of the octic group suggest an almost endless number of exercises with respect to the eight angles under consideration. For instance, the functions of all of them can be expressed in terms of the functions of any other one of them just as easily as in terms of functions of x. Since the octic group has only one invariant operator besides the identity the corresponding angle cannot be used as one of two fundamental angles, while all the other angles besides the one corresponding to the identity can be used as one of these two angles. If x is the angle corresponding to the identity $180°+x$ corresponds to the invariant operator of order 2. If this angle corresponds to the identity while c and p have the same meaning as before the eight operators of the octic group in the third paragraph correspond respectively to the following angles:

$$\begin{array}{cccc} 180°+x & 270°+x & x & 90°+x \\ 270°-x & -x & 90°-x & 180°-x \end{array}$$

Hence, for example,

$$\begin{aligned}
&\sin p^2(180°+x)=\sin x=\cos p(180°+x)=-\sin(180°+x)\\
&\cos p^2(180°+x)=\cos x=-\sin p(180°+x)=-\cos(180°+x)\\
&\tan p^2(180°+x)=\tan x=-\cot p(180°+x)=\tan(180°+x)\\
&\cot p^2(180°+x)=\cot x=-\tan p(180°+x)=\cot(180°+x)\\
&\sec p^2(180°+x)=\sec x=-\csc p(180°+x)=-\sec(180°+x)\\
&\csc p^2(180°+x)=\csc x=\sec p(180°+x)=-\csc(180°+x).
\end{aligned}$$

The use of this octic group makes the work entirely analytic after the functions of the fundamental angles are known. The main interest however lies in the connections which it exhibits and the comprehensive view which it affords. The teacher should be familiar with this view point even if he did not consider it wise to present it to the beginner.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

259. Proposed by ARTEMUS MARTIN, M. A., Ph. D., LL. D., Washington, D. C.

On page 167 of George Bruce Halsted's *Metrical Geometry* (Mensuration), Boston, 1881, Table of Scalene Triangles, is found the following triangle, viz., Sides 21, 61, 65; Area 420. The sides of a rational scalene triangle, whose sides have no common divisor, can not all be odd; one must be even and the other two odd. It is required to find the error in the sides of the above triangle, assuming that the area is correct.

I. Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let a, b, and x be the three sides of the triangle, and m the area, and we have:

$$\tfrac{1}{2}(a+b+x)\times\tfrac{1}{2}(-a+b+x)\times\tfrac{1}{2}(a-b+x)\times\tfrac{1}{2}(a+b-x)=m^2\text{; or}$$

$$[(a+b)^2-x^2][x^2-(a-b)^2]=(4m)^2\text{;}$$

$$x^4-2(a^2+b^2)x^2+(a^2+b^2)^2=4a^2b^2-(4m)^2\text{;}$$

$$\text{hence, } x=\pm\sqrt{\{(a^2+b^2)\pm2\sqrt{[(ab+2m)(ab-2m)]}\}}.$$

As x must be a whole number the radical term must be an exact square, which requires the quantity, $(ab+2m)(ab-2m)$ to be an exact square. Now a and b may be: $a=21$, and $b=61$; or $a=21$, and $b=65$; or $a=61$, and $b=65$.

With $a=61$, and $b=65$, but with none of the other values, the inner radical is an exact square. Reducing the outer radical, we have $x=14$, when $m=420$, if we take the lower sign for the inner radical.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The conditions of the problem require that

$$\tfrac{1}{2}(x+y+z)\tfrac{1}{2}(x+y-z)\tfrac{1}{2}(x-y+z)\tfrac{1}{2}(-x+y+z)=2\times2\times2\times2\times3\times3\times5\times5\times7\times7,$$

$$\text{and } x+y+z=(x+y-z)+(x-y+z)+(-x+y+z).$$

Hence we may have,

$$\left.\begin{array}{r} x+y+z=120,\ 140,\ 150,\ \ldots\ldots\\ x+y-z=\ 70,\ 112,\ \ 98,\ \ldots\ldots\\ x-y+z=\ 42,\ \ 18,\ \ 48,\ \ldots\ldots\\ -x+y+z=\ \ 8,\ \ 10,\ \ \ 4,\ \ldots\ldots \end{array}\right\}\quad \begin{array}{l} x=56,\ 65,\ 73,\ \ldots\ldots\\ y=39,\ 61,\ 51,\ \ldots\ldots\\ z=25,\ 14,\ 26,\ \ldots\ldots \end{array}$$

Hence, the side given as 21 should be 14.

Also solved in part by the following: S. A. Corey, Henry Heaton, A. H. Holmes, L. E. Newcomb, and J. E. Sanders.

260. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The necessary and sufficient condition that a binary form be a perfect nth power is that its Hessian vanish.

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denoting $\frac{\partial u}{\partial x}$ by p, $\frac{\partial u}{\partial y}$ by q, the vanishing of the Hessian shows that $p=f(q)$, *i. e.*, $q=mp$, since both p and q are homogeneous and of the same degree. By Lagrange's method of solving partial differential equations, we have

$$\frac{dx}{m}=\frac{dy}{-1}=\frac{du}{0}.$$

Hence, u=constant, $x+my$=constant, and a general solution is given by

$$u=f(x+my)=(x+my)^n,$$

since u is homogeneous in x, y. It is easily verified that when $u=(x+my)^n$ the Hessian vanishes. Hence this condition is both necessary and sufficient.

II. Solution by the PROPOSER.

A slightly different point of view from the above is afforded by the following method:

The Hessian is the Jacobian of the first derivatives p and q. Hence $p-mq=0$. Also $xp+yq=nu$, n being the order of u. Solving for p and q,

$$p=\frac{nmu}{y+mx},\ q=\frac{nu}{y+mx}.$$

Also, $du=pdx+qdy=nu\dfrac{dy+mdx}{y+mx}$, or $\dfrac{du}{u}=n\dfrac{d(y+mx)}{y+mx}$.

Henge, $\log u=n \log k(y+mx)$, $u=(a_1x+a_2y)^n$.

261. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series, $\frac{1}{n^p}+\frac{3}{n^{2p}}+\frac{5}{n^{3p}}+\frac{7}{n^{4p}}+\frac{9}{n^{5p}}+$.........

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denoting n^{-p} by x, we have

$$\sum_{i=1}^{\infty}\frac{(2i-1)}{n^{ip}}=x[1+3x+5x^2+7x^3+\text{.........}]$$
$$=x\sum(2r+1)x^r=2x\sum rx^r+x\sum x^r$$

$$=2x^2(1-x)^{-2}+x(1-x)^{-1}=x(1+x)(1-x)^{-2}=\frac{n^p+1}{(n^p-1)^2}$$

where we must have $|x|<1$.

Also solved by Henry Heaton, A. H. Holmes, and G. B. M. Zerr.

CALCULUS.

217. Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, Ohio.

Find $\lim\limits_{n\doteq\infty}\dfrac{1}{n}\sqrt[n]{[(n+1)(n+2)\text{.........}(2n)]}$.

I. Solution by the PROPOSER.

Let $x=\lim\limits_{n\doteq\infty}\dfrac{1}{n}\sqrt[n]{[(n+1)(n+2)\text{.........}2n]}$

$$=\lim_{n\doteq\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\text{............}\left(1+\frac{n}{n}\right)}.$$

Then $\log x=\lim\limits_{n\doteq\infty}\dfrac{1}{n}\log\left[\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)\text{............}\left(1+\dfrac{n}{n}\right)\right]$

$$=\lim_{n\doteq\infty}\frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\log\left(1+\frac{\lambda}{n}\right)=\lim_{n\doteq\infty}\frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\left(\frac{\lambda}{n}-\frac{\lambda^2}{2n^2}+\frac{\lambda^3}{3n^3}-\text{.........}\right)$$

$$=\lim_{n\doteq\infty}\sum_{\lambda=1}^{\lambda=n}\sum_{\kappa=1}^{\kappa=n}(-1)^{\kappa-1}\frac{\lambda^\kappa}{\kappa n^{\kappa+1}}.$$

If the method of differences is used for $\sum\limits_{\lambda=1}^{\lambda=n}\lambda^\kappa=1^\kappa+2^\kappa+3^\kappa+\text{.........}$, the κth series of differences is

$$(\kappa+1)^\kappa-\binom{\kappa}{1}\kappa^\kappa+\binom{\kappa}{2}(\kappa-1)^\kappa-\binom{\kappa}{3}(\kappa-2)^\kappa+\text{.....}$$
$$+(-1)^{\kappa-1}\binom{\kappa}{\kappa-1}2^\kappa+(-1)^\kappa 1^\kappa\equiv\kappa!.$$

The $(\kappa+1)$th series is

$$(\kappa+2)^\kappa-\binom{\kappa+1}{1}(\kappa+1)^\kappa+\binom{\kappa+1}{2}\kappa^\kappa-\text{.....}+(-1)^\kappa\binom{\kappa+1}{\kappa}2^\kappa+(-1)^{\kappa+1}1^\kappa\equiv 0,$$

κ being a positive integer.

If the first given number is represented by a and the successive differences by d_1, d_2, ...

$$S_{n,\kappa}=\binom{n}{1}a+\binom{n}{2}d_1+\binom{n}{3}d_2+\text{.....}+\binom{n}{\kappa+1}d_\kappa.$$

First $\lim\limits_{n \doteq \infty} \dfrac{1}{\kappa n^{\kappa+1}} S_{n,\kappa}$ must be sought.

Only in the last term of $S_{n,\kappa}$ n appears in $\kappa+1$ factors, therefore the preceding terms disappear, and

$$\lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa} = \lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} \left[\frac{n(n-1)(n-2).....(n-\kappa)}{(\kappa+1)!} \kappa! \right] = \frac{1}{\kappa(\kappa+1)}.$$

Therefore $\lim\limits_{n \doteq \infty} \sum\limits_{\lambda=1}^{\lambda=n} \sum\limits_{\kappa=1}^{\kappa=n} (-1)^{\kappa-1} \dfrac{\lambda^\kappa}{\kappa n^{\kappa+1}} = \sum\limits_{\kappa=1}^{\kappa=\infty} (-1)^{\kappa-1} \dfrac{1}{\kappa(\kappa+1)}$

$$= \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ... = (1 - \frac{1}{2}) - (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) - .....$$

$$= 2(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...) - 1 = 2 \log 2 - 1 = \log \frac{4}{e}.$$

Hence, $\log x = \log \dfrac{4}{e}$; and $x = \lim\limits_{n \doteq \infty} \dfrac{1}{n} \sqrt[n]{[(n+1)(n+2) .... 2n]} = \dfrac{4}{e}$.

II. Solution by S. A. COREY, Hiteman, Iowa.

Evidently, $\dfrac{1}{n} \sqrt[n]{[(n+1)(n+2) .... (2n)]}$

$$= \frac{1}{n} \sqrt[n]{n^n (1 + \frac{1}{n})(1 + \frac{2}{n}) .... (2)}$$

$$= \sqrt[n]{(1 + \frac{1}{n})(1 + \frac{2}{n}) ....(2)} = s \text{ (say)}.$$

Therefore, $\log s = \dfrac{1}{n} [\log (1 + \dfrac{1}{n}) + \log(1 + \dfrac{2}{n}) + ... + \log 2]$.

Letting $dx = 1/n$, we have,

$$\lim_{n \doteq \infty} \log s = \int_1^2 \log x \, dx = 2 \log 2 - 1, \text{ or } s = \frac{4}{e}.$$

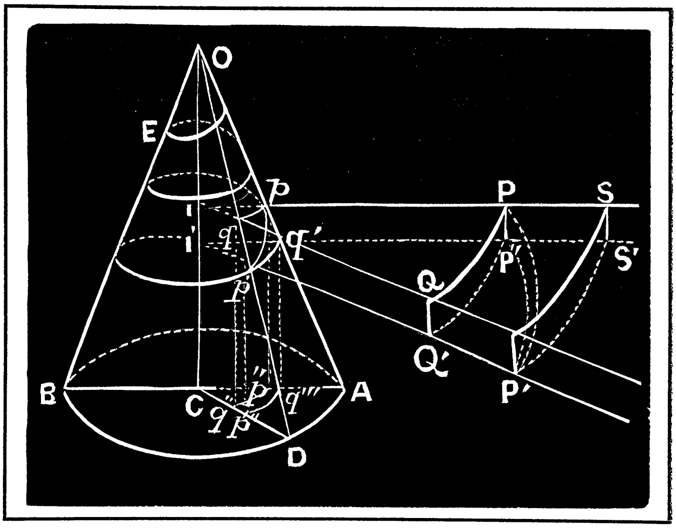
Also solved by Henry Heaton, and J. Scheffer. Several incorrect solutions were received.

209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

Solution by Professor B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Let P and P' be two consecutive positions of the bird at any time; p and p' the two corresponding positions of the end of the string in contact with the cone, it being assumed that the string adheres slightly to the cone in order that the conditions of the problem be fulfilled; $s=pP=OEP$, the length of the string unwound at any time; $Op=w$, $pq'=dw$; $pp'=ds$; $\theta=$ the angle between Ip and the line perpendicular to OC at the beginning of the flight, the angle being measured in the direction in which the bird flys around the cone; $d\theta=$the angle $pIq=$the angle ACD; $s_1=$the distance the bird has flown at any time; $R=$the radius of the base of the cone; and $l=$its slant height. Then



$$ds=[(pq')^2+(q'p')^2]^{\frac{1}{2}} = [(\frac{Rw}{l}d\theta)^2+dw^2]^{\frac{1}{2}} \text{ ....(1).}$$

Since the string passes around the cone n times, it follows that

$$w : R\theta=\frac{l}{n} : 2\pi R, \text{ or } w\frac{l\theta}{2\pi n}, \text{ and } dw=\frac{ld\theta}{2\pi n} \text{ ....(2).}$$

Hence, $ds=\frac{1}{2\pi n}[R^2\theta^2+l^2]d\theta$......... (3), and

$$s=\frac{1}{2\pi n}\int_0^\theta \sqrt{[R^2\theta^2+l^2]}d\theta=\frac{R}{4\pi n}[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})] \text{ .... (4),}$$

where $k=l/R$. If $\theta=2\pi n$, this expression gives the complete length of the string.

Now, $ds_1=PP'=[P'Q'^2+Q'P''^2+QQ'^2]^{\frac{1}{2}}$ ....(5).

But $P''Q'=PQ=PId\theta=(Pp+pI)d\theta=(s+\frac{R\theta}{2\pi n})d\theta$; $Q'P'=pp'=ds$

$$=\frac{R}{2\pi n}\sqrt{(k^2+\theta^2)}d\theta;\text{ and } QQ'=II'=pq'\cos\angle AOC=\frac{\sqrt{(l^2-R^2)}}{l}dw$$

$$=\frac{1}{2\pi n}\sqrt{(l^2-R^2)}d\theta.$$

Hence, by substituting these values of $P'Q'$, $Q'P''$, and QQ' and the value of s from (4) in (5), we have

$$ds_1=\frac{R}{4\pi n}\{[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})+2\theta]^2+4\theta^4+8k^2-4\}^{\frac{1}{2}}\,d\theta.$$

Hence, $$s_1=\frac{R}{4\pi n}\int_0^{2\pi n}\{[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})$$
$$+2\theta]^2+4\theta^2+8k^2-4\}^{\frac{1}{2}}\,d\theta\ldots\ldots\ldots(6).$$

Let $$\theta=\tfrac{1}{2}k\left\{\left[\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right]^x-\left[\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right]^{-x}\right\}$$

$$=\tfrac{1}{2}k[a^x-a^{-x}]=k\sinh(x\log a),\text{ where } a=\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}$$

Hence, when $\theta=0$, $x=0$, and when $\theta=2\pi n$, $x=1$.

Then $$s_1=\frac{l}{4\pi n}\log a\int_0^1\{[\tfrac{1}{2}k^2\sinh 2(x\log a)+k^2\log a^x+2k\sinh(x\log a)]^2$$
$$+4k^2\sinh^2(x\log a)+8k^2-4\}^{\frac{1}{2}}\cosh(x\log a)dx.$$

If we divide the interval (0, 1) into 10 equal parts and find the value of

$$\{[\tfrac{1}{2}k^2\sinh 2(x\log a)+k^2x\log a+2k\sinh(x\log a)]^2$$
$$+4k^2\sinh^2(x\log a)+8k^2-4\}^{\frac{1}{2}}\cosh(x\log a)$$

for each of the values of $x=0$, .1, .2, ..., .9, 1 and if these values be designated by A_0, A_1, A_2, ..., A_{10}, respectively, we have, by Cotes' Method of Approximate Quadrature,*

$$s_1=\frac{l}{4\pi n}\log\left(\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right)$$

$$\times\frac{16067(A_0+A_{10})+106300(A_1+A_9)-48525(A_2+A_8)+272400(A_3+A_7)}{598752}$$
$$+\frac{-260550(A_4+A_6)+427368A_5}{598752}.$$

*See Roger Cotes' *Opera Miscellanea*, p. 33.

The ordinates A_0, A_1, ..., A_{10} may be easily computed by means of a table of Hyperbolic Functions.

In this solution it has been assumed that the unwinding of the string begins at the vertex of the cone. If the unwinding begins at the base of the cone, we replace, in (3), θ by $(2\pi n-\theta)$ and take the negative sign of the radical since it is then a decreasing function of θ. This gives

$$s=\frac{R}{4\pi n}\Bigg[2\pi n\sqrt{(4\pi^2n^2+k^2)}-(2\pi n-\theta)\sqrt{[(2\pi n-\theta)^2+k^2]}$$

$$+k^2\log\left(\frac{2\pi n+\sqrt{(4\pi^2n^2+k^2)}}{2\pi n-\theta+\sqrt{[(2\pi n-\theta)^2+k^2]}}\right)\Bigg].$$

We then have, $ds_1=[(s+\dfrac{(2\pi n-\theta)Rd\theta}{2\pi n})^2+ds^2+\dfrac{l^2-R^2}{4\pi^2n^2}d\theta^2]^{\frac{1}{2}}$ ....(7).

But $ds=-\dfrac{R}{2\pi n}\sqrt{[(2\pi n-\theta)^2+k^2]}d\theta$.

Substituting the values of s and ds in (7) and letting $LR/4\pi n$=the entire length of the string, and $\theta=2\pi n-k\sinh[(1-x)\log a]$, where

$$a=\frac{2\pi n+\sqrt{(4\pi^2n^2+k^2)}}{k},$$

we have,

$$s_1=\frac{R}{4\pi n}\int_0^1\Bigg[\{L-\tfrac{1}{2}k^2\sinh[2(1-x)\log a]+k^2(1-x)\log a$$

$$+2k\sinh[(1-x]\log a]\}^2+4k^2\sinh^2[(1-x)\log a]$$

$$+8k^2-4\Bigg]^{\frac{1}{2}}\cosh[(1-x)\log a]dx,$$

the value of which may be obtained by the foregoing method of approximation.

DIOPHANTINE ANALYSIS.

126. Proposed by **R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.**

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

Remark by DR. L. E. DICKSON, The University of Chicago.

According to A. H. Holmes (MONTHLY, 1905, p. 141), the program could be arranged in 7 evenings, A and B being partners only the first evening. But this solution is clearly erroneous, since by the last condition of the problem A and B shall play against the 15 possible pairs of the remaining six.

I proceed to prove that it is impossible to construct a program of the desired kind. The notation may be chosen so that the order of play for the first evening is

(I) 12, 34, 56, 78.

With 12, 35, must go either 47, 68 or else 48, 67 (since 46, 78 is excluded by I). But these two cases are interchanged by the substitution (78), which does not alter I. Hence, if the problem is possible, there would be a program with I and

(II) 12, 35, 47, 68.

Then with 12, 36 cannot go 47, 58 or 45, 78. In this way we get

(III) 12, 36, 48, 57;
(IV) 12, 37, 46, 58;
(V) 12, 38, 45, 67.

With 13, 24 goes 57, 68 or 58, 67, cases interchanged by the substitution (5768), which leaves I unaltered and permutes II, IV, III, V, cyclically. Hence we may set

(VI) 13, 24, 57, 68;
(VII) 14, 23, 58, 67 (by I and VI);
(VIII) 16, 23, 45, 78 (by III and VII).

Then 15, 23 cannot go with 46, 78; 47, 68; or 48, 67, by VIII, II, VII, respectively. Hence, the problem is impossible.

First modification of problem. If we allow each couple to play exactly three times against every other couple, the problem becomes possible, there being

$${}_8C_2\cdot{}_6C_2\cdot{}_4C_2\cdot{}_2C_2\div 4!=105$$

orders of the players, corresponding to the 105 substitutions of the type (12)(34)(56)(78). While this program is absolutely fair to each player, it would require 105 evenings.

Second modification. Required a program for 35 evenings of duplicate whist between eight players, such that during the series every couple shall play every other couple once and but once, while in each evening there shall be three orders of play in which no two persons play together twice.

Such a program (which is equally fair to all players) is the following:

12, 36; 58, 47	13, 28; 45, 67	12, 35; 46, 78	13, 26; 48, 57
14, 38; 25, 67	15, 27; 38, 46	14, 26; 37, 58	14, 27; 35, 68
15, 26; 34, 78	17, 25; 36, 48	18, 27; 34, 56	16, 23; 45, 78
13, 25; 47, 68	15, 24; 68, 37	12, 34; 57, 68	12, 37; 48, 56
15, 23; 48, 67	16, 28; 35, 47	13, 24; 56, 78	14, 28; 36, 57
16, 24; 38, 57	18, 23; 57, 46	17, 26; 38, 45	18, 24; 35, 67

with cyclic permutations of the last three columns;

12, 67; 35, 48	12, 45; 36, 78	12, 38; 47, 56	13, 27; 45, 68
13, 46; 25, 78	14, 56; 28, 37	17, 23; 46, 58	14, 23; 58, 67
15, 47; 23, 68	17, 68; 25, 34	18, 45; 27, 36	17, 56; 28, 34
13, 58; 26, 47	14, 78; 26, 35	15, 36; 27, 48	16, 27; 34, 58
15, 28; 34, 67	16, 25; 38, 47	16, 37; 24, 58	17, 28; 36, 45
18, 37; 25, 46	17, 58; 24, 36	17, 35; 28, 46	18, 47: 23, 56
16, 34; 27, 58	16, 58; 23, 47	16, 48; 25, 37	
17, 24; 38, 56	17, 34; 28, 56	17, 46; 23, 58	
18, 26; 37, 45	18, 25; 37, 46	18, 36; 27, 45	

This solution, which I made in 1905, has been re-checked at two different times. Note that the first 24 orders furnish a Thompson program for 24 evenings, each couple to play against the other three couples in turn. It would be interesting to know whether or not 24 is a maximum* in the Thompson problem.

134. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

How many sets of solutions has the congruence $\lambda+\mu+\nu+\xi\equiv 0 \pmod{p-1}$ p being a prime number; the order of λ, μ, ν, ξ being disregarded.

Solution by the PROPOSER.

Assume that $p>5$, and let n_i be the number of solutions in which i of the λ, μ, ν, ξ are congruent to each other (mod $p-1$). If $p\equiv 1$ (mod 4), then $n_4=4$, viz.,

$$\lambda\equiv\mu\equiv\nu\equiv\xi\equiv 0,\ \frac{p-1}{4},\frac{p-1}{2},\frac{3(p-1)}{4}.$$

If $p\equiv 3$ (mod 4), $n_4=2$, viz., $\lambda\equiv\mu\equiv\nu\equiv\xi\equiv 0,\ \frac{p-1}{2}$.

Next, let $i=3$, so that the congruence reduces to $\lambda+3\nu\equiv 0 \pmod{p-1}$.

*Note that a Thompson program for 22 evenings is given by (I)–(VIII) and

13, 25, 46, 78;	13, 26, 47, 58;	13, 27, 48, 56;	14, 25, 37, 68;
14, 26, 38, 57;	15, 24, 36, 78;	15, 27, 34, 68;	16, 28, 34, 57;
17, 24, 38, 56;	17, 26, 35, 48;	17, 28, 36, 45;	18, 23, 46, 57;
18, 24, 35, 67;	18, 25, 36, 47;	18, 26, 37, 45.	

These 22 orders are mutually consistent, while no other is consistent with them.

Evidently ν cannot be congruent to $0, \frac{p-1}{4}, \frac{p-1}{2}$ or $\frac{3(p-1)}{4}$, for then $\nu \equiv \lambda$. Otherwise ν can have any value $(\bmod\, p-1)$, so that when $p \equiv 1 \pmod 4$, $n_3 = p-1-4 = p-5$, and when $p \equiv 3 \pmod 4$, $n_3 = p-3$.

When $i=2$, there are two possibilities.

(1) Let the parameters be congruent in pairs so that

$$2\lambda + 2\nu \equiv 0 \pmod{p-1}.$$

Here, if $\nu \equiv 0$, then must $\lambda \equiv \frac{p-1}{2}$ and inversely. Let all values $(\bmod\, p-1)$ be assigned to ν in succession. With each value of ν belongs $dv(2; p-1) = 2$ values of λ determined by the congruence; as ν assumes its values λ repeats these values in another order twice. Hence, removing the two excluded solutions $(0, 0)$, $\left(\frac{p-1}{2}, \frac{p-1}{2}\right)$, we have, if $p \equiv 1 \pmod 4$,

$$n_{2_1} = \frac{2(p-1)}{2} - 2 = p-3,$$

and if $p \equiv 3 \pmod 4$,

$$n_{2_1} = \frac{2(p-1)}{2} + 1 - 2 = p-2.$$

(2) Next suppose that two parameters are congruent and the other two incongruent, so that

$$\lambda + \mu + 2\nu \equiv 0 \pmod{p-1}.$$

For the two values $\nu \equiv 0$, $\nu \equiv \frac{p-1}{2}$ the congruence reduces to $\lambda + \mu \equiv 0 \pmod{p-1}$. Then μ is not $\equiv 0$ or $\frac{p-1}{2}$, since then $\lambda \equiv \mu$. Apart from these two exceptions μ can have any value $(\bmod\, p-1)$, and for each μ a λ is determined by the congruence. Thus the cases $\nu \equiv 0, \frac{p-1}{2}$ yield $\frac{2(p-3)}{2}$ solutions. Aside from the two exceptions given ν can have any value $(\bmod\, p-1)$. We may now assign to μ $p-3$ values exclusive of the value ν_1 assigned to ν, and the value $p-1-3\nu_1$. Then λ receives all values $(\bmod\, p-1)$ exclusive of the two $p-1-3\nu_1$ and ν_1, respectively. Two values of λ are congruent to μ and must be excluded; *e. g.*, $\lambda \equiv \mu \equiv \frac{p-1-2\nu_1}{2}$ and $\lambda \equiv \mu \equiv p-1-\nu_1$. Aside from these, λ's values are the values of μ in another order. Hence, since $(\lambda\, \mu\, \nu)$ and $(\mu\, \lambda\, \nu)$ constitute the same solution, we obtain $(p-3)\left(\frac{p-3}{2} - 1\right) = (p-3)\left(\frac{p-5}{2}\right)$ new solutions. Thus $n_{2_2} = p-3+(p-3)\left(\frac{p-5}{2}\right) = \frac{1}{2}(p-3)^2$.

Now if we have regard to the order of λ, μ, ν, ξ, there are $(p-1)^3$ ways of satisfying the original congruence. For each of three parameters can be assigned in $p-1$ ways, and the congruence determines the remaining one. Hence we have

$$(p-1)^3=n_4+{}_3C_4n_3+6n_{2_1}+2{}_2C_4n_{2_2}+{}_4P_4n_1,$$
$$(p-1)^3=n_4+4n_3+6n_{2_1}+12n_{2_2}+24n_1.$$

Substituting the values found above for n_{2_2}, n_{2_1}, n_3, n_4, we get, when $p\equiv1 \pmod 4$,

$$n_1=\tfrac{1}{24}(p^3-9p^2+29p-21)$$

and when $p\equiv3 \pmod 4$,

$$n_1=\tfrac{1}{24}(p^3-9p^2+29p-33).$$

Hence, when $p>5$ is a prime of the form* $4l+1$ the number (N) of sets of solutions of $\lambda+\mu+\nu+\xi\equiv0 \pmod{p-1}$ is,

$$N=\tfrac{1}{24}(p^3-9p^2+29p-21)+\tfrac{1}{2}(p-3)^2+p-3+p-5+4$$
$$=\tfrac{1}{24}(p^3+3p^2+5p-9).$$

But when p is of the form $4l+3$,

$$N=\tfrac{1}{24}(p^2-9p^2+29p-33)+\tfrac{1}{2}(p-3)^2+p-2+p-3+2$$
$$=\tfrac{1}{24}(p^3+3p^2+5p+3).$$

GEOMETRY.

284. Proposed by JOHN JAMES QUINN, Ph. D., Warren, Pa.

a) Suppose that two radii R and r, whose center is the origin, revolve with uniform angular velocities 3θ and θ, respectively. What is the equation of the locus of P, the projection parallel to the X axis of the extremity of the radius r on the radius R produced if necessary.

b) Apply this curve to the trisection of an angle.

c) Suppose the ratio of their velocities is $n\theta:\theta$. Show how we can effect the multisection of an angle.

Solution by A. H. HOLMES, Brunswick, Maine.

a) Take O, the center of the circle, radius a, as the origin of coördinates. Then taking any angle θ, we shall have $r\sin3\theta=a\sin\theta$.

$\therefore r=\dfrac{a\sin\theta}{\sin3\theta}$ is the equation of the locus of the point P.

b) Construct the curve $r=\dfrac{a\sin\theta}{\sin3\theta}$. On the circumference of the circle

*The values $p=3$, 5 constitute exceptions in the method employed. The final result does not hold for $p=5$. But, by inspection, when $p=5$, $N=10$; and when $p=3$, $N=3$.

take an arc equal to any angle ψ from axis of x. Draw a line from the upper limit of the arc to the origin. From the point P, where this line cuts the curve $r=\dfrac{a\sin\theta}{\sin 3\theta}$, draw a line parallel to axis of x cutting the given arc. From this point draw a line to O making an angle$=\frac{1}{3}\psi$.

c) The equation of the locus of P would be $r=\dfrac{a\sin\theta}{\sin n\theta}$, and the multisection would be similar to the trisection.

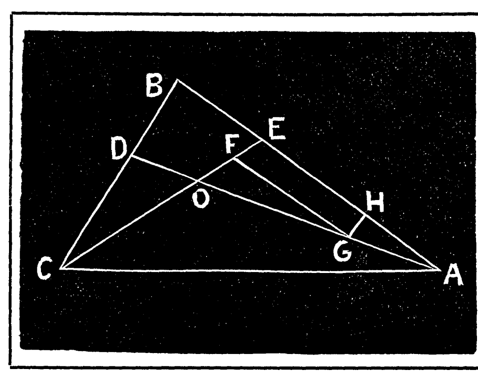
NOTE. By projecting parallel to the y-axis Dr. Zerr obtains in a similar way the locus $r=\dfrac{a\cos\theta}{\cos n\theta}$, and effects the n-section of the angle in a manner similar to the above. ED.

285. Proposed by G. E. BROCKWAY, Nashua, N. H.

Prove without the aid of the circle, that if the bisectors of the angles of a triangle be drawn, the greatest bisector falls on the least side.

I. Solution by ALFRED H. PARROTT, North Dakota Agricultural College, N. D.

Given scalene triangle ABC, and bisectors of angles A and C, supposing $\angle C>\angle A$. If $\angle C>\angle A$, then side $AB>$side BC, and we are to prove bisector AD on side $BC>$bisector CE on side AB. If $\angle C>\angle A$, $\angle OCA>\angle OAC$ [If wholes are unequal, then halves, etc.].



$\therefore AO>OC$ [If angles of a triangle, etc.]. Now if $OD>$ or $=OE$, the proposition is evident. But suppose $OD<OE$. Then on OE take $OF=OD$, and on OA take $OG=OC$, and draw FG. Then draw GH parallel to CE.

$\triangle FOG=\triangle COD$ [Three sides on one equal respectively, etc.]. Then $\angle FGO=\angle DCO>\angle EAO$ [Halves of unequals, etc.].

Consider $\triangle FGO$ and $\triangle EAO$; $\angle O\equiv\angle O$, $\angle FGO>\angle EAO$ [Previous proof]. Therefore $\angle GFO<\angle AEO$.

A line drawn parallel to FG and through E will then intercept HG between H and G and HG is therefore $>EF$.

Now $\angle AHG>\angle HAG$, for $\angle AHG=\angle AEC$, [sides respectively parallel], $\angle AEC=\angle B+\angle ECB$ [Exterior angle of a triangle equals sum of, etc.]. But $\angle HAG<\angle ECB$, and therefore $<\angle B+\angle ECB$, and hence $<\angle AHG$.

$\therefore AG>HG>EF$, or $AO-GO>OE-OF$,

$AO+OF>OE+GO$, $AO+OD>EO+OC$.

Hence $AD>EC$. Q. E. D.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let in a triangle ABC $AB>AC$, and BE and CD be the bisectors of angles B and C of the triangle, cutting AC and AB in E and D, respectively. To prove $CD<BE$.

Through E draw EH parallel to CB. Draw HI parallel to AC, and HK parallel to DC, cutting AC in K. It is evident that $HE=HB$, and $HE=EK$; moreover, $\angle HIB=\angle ACB>\angle HBI$. Therefore, $BH>HI$, and hence $EK>HI$ or EC. The point K, therefore, lies on AC produced, and hence, the point D between A and H. Comparing the two triangles BHE and HEK, we see at once that $BE>HK$. But $CD<HK$, *a fortiori*. Therefore $CD<BE$.

Also solved by Henry Heaton, A. H. Holmes, Rev. J. H. Meyer, and G. B. M. Zerr.

286. Proposed by S. F. NORRIS, Baltimore City College, Baltimore. Md.

On the sides of a given triangle measure off equal distances from the extremities of the base, and at these points erect perpendiculars to the sides. Find the locus of the point of intersection of these perpendiculars. Solve by methods of analytic geometry.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ACB be the triangle, AB the base. Lay off $AE=BD=d$ on the two sides AC, BC, and erect perpendiculars at E and D cutting each other at H, and CB and AC at F and G, respectively. Then $CE=b-d$, $CF=(b-d)\sec C$.

Hence $\dfrac{x}{(b-d)\sec C}+\dfrac{y}{b-d}=1$, is the equation to EF..........(1).

Also $CD=a-d$, $CG=(a-d)\sec C$, and $\dfrac{x}{a-d}+\dfrac{y}{(a-d)\sec C}=1$, is the equation to DG..........(2). Now (1) and (2) may be written as follows:

$$x\cos C+y=b-d \;........\; (3),$$
$$x+y\cos C=a-d \;.........\; (4).$$

Eliminating d we get $x-y=\dfrac{a-b}{1-\cos C}=\frac{1}{2}(a-b)(\operatorname{cosec}\frac{1}{2}C)^2$, as the locus of H, the intersection required.

Also solved by G. W. Greenwood, Henry Heaton, A. H. Holmes, and J. Scheffer.

287. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill

Show that the points whose abscissae are 0, $a\sqrt{3}$, and $-a\sqrt{3}$ are points of inflexion on the locus $x^2y-a^2x+a^2y=0$.

Solution by the PROPOSER.

Let P be the point whose abscissa is $a\sqrt{3}$ and whose ordinate is therefore $\dfrac{a\sqrt{3}}{4}$. Let Q be any point on the curve. The coördinates of Q are, therefore,

$$a\sqrt{3}+r\cos\theta,\quad \frac{a\sqrt{3}}{4}+r\sin\theta,$$

where $PQ=r$, and the line PQ makes an angle θ with the x-axis. Hence, since Q lies on the curve, we have

$$2ar(\cos\theta+8\sin\theta)+\sqrt{3}\,ar^2\cos\theta(\cos\theta+8\sin\theta)+4r^3\cos^2\theta\sin\theta=0.$$

One value of r is zero for all values of θ; hence one branch of the curve passes through P. *Two* more values of r are zero, when $8\sin\theta+\cos\theta=0$. Hence P is a point of inflexion. In a similar manner we can show that the other points named are points of inflexion.

Also solved by A. H. Holmes, J. Scheffer, and G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

265. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Obtain the reduced cubic $4\theta^3-I\theta+J=0$ of the biquadratic $ax^4+4bx^3+6cx^2+4dx+e=0$.

266. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

Find the nth term and the sum of n terms of the series $1+3+7+17+\ldots$

267. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the trigonometric functions of x as infinite continued fractions.

CALCULUS.

221. Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, Ohio.

If a, b, c, ... represent all the prime numbers 2, 3, 5, ... prove that

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\ldots=\frac{15}{\pi^2}.$$

222. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Evaluate $\int_0^1(1+x^m)^n\log x\,dx$.

DIOPHANTINE ANALYSIS.

137. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Prove that all multiply perfect numbers of multiplicity n having only n distinct primes are comprised in $n=2, 3, 4$.

GEOMETRY.

290. Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given nine points lying by threes in three columns and in three rows, draw through them, by continuous motion, a broken line composed of only four straight segments, and passing but once through each of the nine points. [A current puzzle.]

291. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

The pedal line of any point on a triangle's circum-circle bisects the distance between this point and the ortho-center of the triangle.

292. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Apply the locus of $(x^2+y^2)^3=mx^3$ to the problem of finding a cube m times a given cube.

293. Proposed by W. J. GREENSTREET, M. A., Editor Mathematical Gazette, Stroud, England.

A variable circle touches an ellipse, and the chord of contact through the other two points of intersection touches a similar coaxial ellipse. Find the locus of the center of the variable circle.

294. Proposed by JOHN JAMES QUINN, Ph. D., Warren, Pa.

a) Suppose an indefinite line be pivoted at the end of a revolving radius whose center is the origin; and the initial position of the radius is coincident with the X-axis and the pivoted line perpendicular to it. As the radius revolves through equal amounts of arc the line moves to the right over corresponding equal intercepts on the X-axis. What is the equation of the locus of a point on the line whose distance from the end of the radius is equal to a diameter?

b) Show how the locus can be applied to the multisection of an angle.

c) Suppose the diameter be laid off in both directions.

MECHANICS.

190. Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905).

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A pole hinged at the bottom leans against the mid-point of a smooth rope suspended from two supports of equal height. Determine the position of equilibrium.

192. Proposed by REV. J. H. MEYER, S. J., College of the Sacred Heart, Augusta, Ga.

Find the velocity of a planet at a given point in its orbit.

NOTES AND NEWS.

Dr. C. N. Haskins has been appointed assistant professor of mathematics in the University of Illinois.

Professor J. H. Tanner will return to his work at Cornell University this year, after a year's leave of absence.

At Cornell University Mr. W. H. Carruth has been appointed fellow, and Mr. C. F. Craig assistant in mathematics.

Professor S. E. Slocum, of the University of Illinois, has accepted the professorship of applied mathematics at the University of Cincinnati.

Mr. C. E. Colpitts, formerly assistant in mathematics at Cornell University, has been appointed adjunct professor of mathematics in the Georgia School of Technology.

Columbia University has received $5,000 with which to establish a mathematical prize, in memory of John D. Van Buren, Jr., a member of the class of 1903. The fund is the gift of Mrs. Louise T. Hoyt.

A movement is being promoted to organize a new section of the American Mathematical Society, with headquarters at St. Louis or Kansas City. It is hoped to bring into more active participation in the business of the society a large number of its members who are situated at a distance from any present section.

The following persons have been elected members of the American Mathematical Society: Rev. R. D. Carmichael, Hartselle, Ala,; Mr. F. L. Griffin, University of Chicago; Mr. W. R. Longley, University of Chicago; Mr. W. D. MacMillan, University of Chicago; Mr. F. W. Owens, Evanston Academy; Dr. J. J. Quinn, High School, Warren, Pa.; Mr. W. J. Risley, University of Illinois; Dr. R. P. Stephens, Wesleyan University; Mr. J. D. Suter, Iowa State College; Mr. A. M. Wilson, McKinley High School, St. Louis, Mo. The total membership of the society is now five hundred and thirty.

The University of Chicago announces the following advanced courses in mathematics for the summer quarter, June 19-September 1: By Professor O. Bolza: Elliptic functions, four hours; Functions of a real valuable, four hours. By Professor H. Maschke: Geometry, four hours. By Professor H. E. Slaught: Elliptic integrals, four hours; Analytical geometry, five hours. By Professor L. E. Dickson: Algebraic analysis, four hours; Theory of substitutions, four hours. By Dr. A. C. Lunn: Integral calculus, five hours; General seminar, two hours. By Mr. N. J. Lennes: Pedagogy of mathematics, four hours.

Professor L. E. Dickson will offer, in addition to his advanced courses, a course in the correspondence study department in Plane Trigonometry, by the Laboratory Method.

Mr. William Marshall, of the University of Michigan, will spend the coming year in study abroad.

Professor G. A. Miller has been called to an associate professorship of mathematics at the University of Illinois.

Professor E. L. Richards, of Yale University, will retire from active service at the close of the present academic year.

Mr. Clarence M. Thorne has resigned his instructorship in the State University of Iowa. The position will be filled by appointment in June.

The current *Bookman* contains a contribution by Professor C. J. Keyser, of Columbia University, entitled "Concerning Research in American Universities."

Dr. Saul Epsteen, formerly instructor, has been promoted to an assistant professorship in mathematics in the University of Colorado, being now second in rank to the head of the department.

Professor Oskar Bolza, of the University of Chicago, who is spending the year in travel and study in Central Europe and Egypt, will resume his lectures at the university at the beginning of the summer quarter, June 19, 1906.

Dr. Samuel Hart Wright (M. A., M. D., Ph. D.), a versatile scholar in many fields, died at his home in Penn Yan, N. Y., on October 7, 1905, at the age of eighty years. Dr. Wright is known to the scientific world through the publication in 1848-50 of his astronomical tables, and for his studies in botany and geology. He had been a reader the MONTHLY since it was founded in 1894.

ERRATUM.

On page 85, Vol. XIII, No. 4, the term involving B_3 should have been extended as .000,000,276 instead of .000,000,009, and the final value of the integral should read 1.657,636,257 instead of 1.657,636,524.

By special arrangement, the paper by Miss McKelden appears as a supplement to the present number of the MONTHLY. It is the policy of the Editors to devote the regularly available space to elementary papers of general interest, and not to technical papers, however great their value.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. JUNE-JULY, 1906. NOS. 6-7.

GROUPS OF ORDER 2^m THAT CONTAIN CYCLIC SUBGROUPS OF ORDER 2^{m-3}.

By ALICE M. McKELDEN.

INTRODUCTION.

A determination of groups of order p^3, and p^4 has been given by Hölder;* another determination of groups of orders p^3 and p^4, by Young.† The distinct types of groups of orders p^2, p^3, p^4 have been constructed and tabulated by Burnside.‡ Types of groups of order p^5, in addition to those of order p^3, p^4, which were considered first to illustrate the method of treatment, have been determined by Bagnera;§ and types of groups of order p^6 have been determined by Potron.¶ The number of groups of order p^m, which contain self-conjugate cyclic subgroups of orders p^{m-1} and p^{m-2}, respectively, has been discussed by Burnside;‖ the number of groups of order p^m that contain cyclic non-self-conjugate subgroups of order p^{m-2} has been determined by Miller;** and the groups of order p^m, which contain cyclic subgroups of order p^{m-3} (p odd prime) have been determined by Neikirk.††

*_Mathematische Annalen_, Vol. 43 (1893), pp. 301—412.

†_American Journal of Mathematics_, Vol. 15 (1893), pp. 124—178.

‡_Theory of Groups of a Finite Order_, pp. 81—89.

§_Annali di Mathematica_, Vol. 3 (1898), pp. 137—228; 263—275.

¶ _These_ (1904), Gauthier Villars, Paris.

‖ _Loc. Cit._ pp. 75—81. One of the groups has been omitted by Burnside and XI~XII, p. 81. See "A Note on Groups of Order 2^m, which contain Self-Conjugate Subgroups of Order 2^{m-2}," Hallett, _Science_, New Series, Vol. 21, No. 527, Feb. 3, 1905.

**_Transactions American Mathematical Society_, Vol. 2 (1901), p. 259, and Vol. 3 (1902), p. 383.

††Publications of The University of Pennsylvania, Mathematics, No. 3. _Transactions of American Mathematical Society_, Vol. 6 (1905), No. 3.

The object of this paper is to treat in detail the construction of groups of order 2^m $(m>6)$ that contain cyclic subgroups of order 2^{m-3}, and to give a complete tabulation of such groups.

The treatment is based on the following division of the groups, where P is an operator of G_m of maximum order:

I. P is of order 2^m; II. P is of order 2^{m-1}; III. P is of order 2^{m-2}; IV. P is of order 2^{m-3}. The group G_m contains a series of subgroups G_{m-1}, G_{m-2}, ..., G_{m-r}, of order 2^{m-1}, 2^{m-2}, ..., 2^{m-r}, containing any G_{m-r}; and each one is self-conjugate in the one preceding.*

I, II, III. GROUPS CONTAINING P, OF ORDER 2^m, 2^{m-1}, OR 2^{m-2}.

These groups have all been determined and tabulated by Miller† and Burnside,‡ and the thirty-three types are given here for reference.

I. Abelian group of the type (m).

II. $P^{2^{m-1}}=1$.

$$Q^{-1}PQ=P^{\omega+2^{m-2}\kappa},\quad Q^2=P^{2^{m-2}\lambda_1};\quad \omega=\pm1,\ \kappa=0,\ 1,\ \lambda_1=0;\ \omega=-1,\ \kappa=0,\ \lambda_1=1.$$

III. $P^{2^{m-2}}=1$.

$$Q^{-1}PQ=P^{\omega+2^{m-4}\beta_1},\ Q^4=P^{2^{m-3}\lambda_1};\ \omega=\pm1,\ \beta_1=0,1,2,\ \lambda=0;\ \omega=-1,\ \beta_1=0,\ \lambda_1=1.$$

$$Q^{-1}PQ=Q^2P^{1-2^{m-4}\kappa},\quad Q^4=1,\quad Q^{-2}PQ^2=P^{1+2^{m-3}\kappa},\quad \kappa=0,\ 1.$$

$$Q^{-1}PQ=Q^2P^{-1+2^{m-3}\beta_2},\quad Q^4=1,\quad Q^{-2}PQ^2=P^{1+2^{m-3}\kappa},\quad \beta_2=0,\ 1,\ \kappa=0,\ 1.$$

$$Q^{-1}PQ=Q^2P^{-1},\quad Q^4=P^{2^{m-3}},\quad Q^{-2}PQ^2=P^{1+2^{m-3}\kappa},\quad \kappa=0,\ 1.$$

$$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2},\quad Q^4=P^4,\quad \beta_2=0,\ 1,\ \kappa=0;\ \beta_2=0,\ \kappa=1.$$

$$Q^{-1}PQ=P^{1+2^{m-3}\kappa},\quad R^{-1}PR=P^{\omega_1+2^{m-3}\beta_1},\quad R^{-1}QR=QP^{2^{m-3}b_1},\quad Q^2=1,\quad R^2=1,$$
$$\omega_1=\pm1,\ \kappa=0;\ \beta_1=0,1,\ b_1=0;\ \beta_1=0,\ b_1=1;\ \omega_1=-1,\ \kappa=\beta_1=1,\ b_1=0,\ 1.$$

$$Q^{-1}PQ=P,\quad R^{-1}PR=P^{-1},\quad R^{-1}QR=Q,\quad Q^2=1,\quad R^2=P^{2^{m-3}}.$$

IV. GROUPS CONTAINING P, OF ORDER 2^{m-3}.

The eighth power of every operator is in $\{P\}$. The groups may be divided into three classes.

Class 1. An operator Q of G_m may be so chosen that Q^4 is not contained in $\{P\}$.

Class 2. The fourth power of every operator is in $\{P\}$ and there is an operator Q, of G_m, such that Q^2 is not in $\{P\}$.

Class 3. The second power of every operator is contained in $\{P\}$.

These classes will be treated in order in Parts 1, 2, and 3, respectively.

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*Burnside, *Loc. Cit.*, Art. 55, p. 65.

†Miller, *Transactions American Mathematical Society*, Vol. 2 (1901), p. 259, and Vol. 3 (1902), p, 383.

‡Burnside, *Loc. Cit.*, pp. 75—81.

PART 1. Q^4 IS NOT IN $\{P\}$.

In this case $Q^8=P^{8\lambda}$, and G_m is generated by P and Q, since there are 2^m distinct operators of the form $Q^\alpha P^\beta$ ($\alpha=0\ldots 7$, $\beta=0\ldots 2^{m-3}-1$). Also $G_{m-3}=\{P\}$, and G_{m-2} is generated by P and some other operator of G_m, $Q^\alpha P^\beta$. Hence G_{m-2} contains Q^α. So $G_{m-2}=\{P, Q^\alpha\}=\{P, Q^4\}$. Likewise $G_{m-1}=\{P, Q^2\}$.

In G_{m-2}, $Q^{-4}PQ^4=P^{\omega+2^{m-4}\kappa}$; in G_{m-1}, $Q^{-2}PQ^2=Q^{4\alpha}P^\beta$, and in G_m, $Q^{-1}PQ=Q^{2a}P^b$. Three cases arise: (A) $\alpha=a=0$; (B) $\alpha=0$, $a=1, 2, 3$; (C) $\alpha=1$, $a=1, 2, 3$. We shall subdivide these cases where necessary, using $A_1, A_2, \ldots$ to distinguish the 1st, 2nd, ... subcases of A, etc.

(A) $\alpha=a=0$. Here $\{P\}$ is self-conjugate in G_m.

$$Q^{-1}PQ=P^b=P^{\omega_1+2^{m-6}b_1}\ (\omega_1=\pm 1,\ b_1=0\ldots 7)\ldots(1).$$

Let $b_1=2^n b_2$, where b_2 is odd and $n=0\ldots 3$. In general, $(R^zQ^yP^xR^v\ldots)^s$ will be represented by $[z, y, x, v\ldots]^s$; also $\frac{1+(-1)^{s-1}}{2}$ by θ_s, and $\frac{1+(-1)^s}{2}$ by ϕ_s. From (1),

$$[0, -y, x, 0, y]=[0, 0, x(\omega_1+2^{m-6+n}b_2)^y]\ldots(2).$$

$$[0, y, x]^2=[0, 2y, 2x\{\frac{1+(\omega)^y}{2}+(\omega)^{y-1}2^{m-7+n}b_2y\}]\ldots(3).$$

Let $Q'=QP^x$; then $Q'^8=1$, if x be chosen to satisfy

$$\lambda+x(1+\omega_1)/2+2^{m-7+n}b_2x\equiv 0 \pmod{2^{m-6}}\ldots(4),$$

where for $\omega_1=-1$, $\lambda=2^{m-7}\lambda_1$. This is always possible for $\omega_1=1$, except when $m=7$; b_1 and λ odd, when $Q^8=P^8$; and for $\omega_1=-1$, when b_1 is odd. For $\omega_1=-1$, and b_1 even, $Q'^8=P^{2^{m-4}\lambda_1}$; $\lambda_1=0, 1$. In this last case,

$$[0, y, x]^s=[0, sy, x(s\phi_y+\theta_{sy})+2^{m-7+n}b_2xy\{s-s^2\phi_y-(2s-1)\theta_{sy}\}]\ldots(5),$$

and the groups, where $n=2$, correspond to those where $n=3$, for $\lambda_1=1$. The correspondence is given by $C=\begin{bmatrix}Q^4P, & Q\\ P, & Q\end{bmatrix}$.

Let $Q'=Q^y$; then $Q'^{-1}PQ'=P^{\omega_1+2^{m-6+n}}$, if y be chosen to satisfy

$$b_2y+\frac{y(y-1)}{2}2^{m-6+n}b_2^2+\ldots\equiv 1 \pmod{2^{3-n}}\ldots(6),$$

which choice is always possible. So in (A) there are ten types. They are given by the following defining relations:

$Q^{-1}PQ=P^{\omega_1+2^{m-6+n}}$, $Q^8=P^{2^{m-4}\lambda_1}$, $P^{2^{m-3}}=1$; $\lambda_1=0$; $\omega_1=\pm 1$, $n=1, 2, 3$;
$m>7$, $\omega_1=\pm 1$, $n=0$. $m=7$, $\omega_1=-1$, $n=0$; $\lambda_1=1$, $\omega_1=-1$, $n=1, 3$.

$Q^{-1}PQ=P^{1+2}$, $Q^8=P^8$, $P^{2^4}=1$.

(B) $\alpha=0$, $a=1, 2, 3$.

$$Q^{-1}PQ=Q^{2a}P^{b}\ldots(1),$$

$$Q^{-2}PQ^{2}=P^{\omega_1+2^{m-5}\beta_1}\ (\omega_1=\pm1,\ \beta_1=0\ldots3)\ldots(2).$$

From (2), $[0,\ -2y_1,\ x,\ 0,\ 2y_1]=[0,\ 0,\ x(\omega_1+2^{m-5}\beta_1)^{y_1}]\ldots(3)$. Hence

$$[0,\ 2y_1,\ x]^{s}=[0,\ 2sy_1,\ \theta_s x\{1+\omega_1 2^{m-5}\beta_1 y_1(s-\theta_s)\}+(s-\theta_s)\ \{\frac{1+(\omega_1)^{y_1}}{2}$$
$$+(\omega_1)^{y_1-1}2^{m-6}\beta_1 y_1]x]\ldots(4).$$

Transform (1) by Q and place equal to (2); raise to the power 8λ. There result $b=1+2b_1$, and the congruences

$$a(1+b_1)\equiv 0\ (\mathrm{mod}\ 2)\ldots(5);$$

$$\lambda[4a\lambda-1+2b_1+(\omega_1)^{a}(1+2b_1)]\equiv 0\ (\mathrm{mod}\ 2^{m-5})\ldots(6),$$

$$4a\lambda(1+b_1)+(1+2b_1)[1+b_1(1+(\omega_1)^{a})]+2^{m-5}\beta_1[(\omega_1)^{a-1}ab_1+1]$$
$$\equiv\omega_1\ (\mathrm{mod}\ 2^{m-3})\ldots(7).$$

For $\omega_1=-1$, a and b are odd by (8). Hence P is of an order lower than 2^{m-3}, and there are no groups in this case.

For $\omega_1=1$, from (1) and (4),

$$[0,\ -y,\ x,\ 0,\ y]=[0,\ 2ax\theta_y,\ x\{1+2^{m-6}\beta_1(y-\theta_y)\}\{1+2\theta_y[b_1+2^{m-7}a\beta_1(1+2b_1)]\}$$
$$+2^{m-6}a\beta_1\theta_y\{2\lambda x(y-1)+\theta_x(1+2b_1)(2x-3)\}]\ldots(8),$$

and $[0,\ y,\ x]^{2}=[0,\ 2(y+ax\theta_y),\ x+x\{1+2^{m-6}\beta_1(y-\theta_y)\}\{1+2\theta_y[b_1+2^{m-7}a\beta_1$
$$\times(1+2b_1)]\}+2^{m-6}a\beta_1\theta_y\{2\lambda x(y-1)+\theta_x(1+2b_1)(2x-3)\}]\ldots(9).$$

(B_1) $a=2$.

(a) $b_1=2^{m-7}b_2$; (b) $b_1=-1+2^{m-7}b_2$; $(b_1=0\ldots7)$. Let $Q'=QP^{-\lambda}$. Then, for (a), $Q'^{8}=1$; for (b), $Q'^{8}=P^{2^{m-4}\lambda_1}$ $(\lambda_1=0,\ 1)$. $(b)\sim(A)$ with $C=\begin{bmatrix}Q^2P, & Q\\ P, & Q\end{bmatrix}$. $(a)\sim(a_1)$, (a_2), where $b_2=\beta_1=0,\ 2$, respectively; and (a_3) where $b_2=\beta_1=1$, for $m>8$; and $b_2=3$, $\beta_1=1$, for $m=8$. $C=\begin{bmatrix}P, & Q^{y'}Q^{2^{m-7}x_1'}\\ P, & Q\end{bmatrix}$, where $y'=1+2y'_1$ and y'_1 and x'_1 are determined to satisfy the congruences $\beta_1(1+2y'_1)\equiv\beta'_1(\mathrm{mod}\ 4)$; $\lambda_1+x'_1\equiv\lambda'_1$ (mod 2); and $2y'_1(\beta_1-2\lambda_1)-2x'_1\equiv b'_2-b_2$ (mod 8). Of these (a_3) for $m=8$, and (a_2) for $m=7\sim(A)$ with $C=\begin{bmatrix}QP, & QP^4\\ P, & Q\end{bmatrix}$ and $\begin{bmatrix}QP, & QP^2\\ P, & Q\end{bmatrix}$, respectively. Hence there are three types for $m>8$, two for $m=8$, and one for $m=7$.

(B_2) $a=1,\ 3,\ m>7$. From (5), $b_1=1+2b_2$. Let $Q'=QP^{-\lambda}$ for λ even. Then $Q'^{8}=P^{2^{m-4}\lambda_1}$ $(\lambda_1=0,\ 1)$. Let $Q'=Q^{y}$ (y odd), for λ odd. Then $Q'^{8}=P^{8}$, if y be determined to satisfy $\lambda y\equiv 1\ (\mathrm{mod}\ 2^{m-6})$.

(a) For $\lambda=2^{m-7}\lambda_1$, from (7) $b_2=-1+2^{m-7}b_3$ ($b_3=0\ldots3$). (a)$\sim$the seven types $a=1$; $b_3=\beta_1=0$, 1, 2, $\lambda_1=0$, 1; $b_3=2$, $\beta_1=0$, $\lambda=0$; with $C=\begin{bmatrix} Q^{2y_1}P^x, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where $x=1+2x_1$, and the variables satisfy $a_1-a_1'\equiv x_1+y_1+x_1'$ (mod 2); $\beta_1(1+2x_1')\equiv\beta_1'$ (mod 4); and $\lambda_1\{a_1-a_1'+x_1+y_1-x_1'+2[a_1(x_1-x_1')-a_1'x_1']\}-\beta_1\{1-x_1-x_1'-b_3'y_1+2[a_1(x_1+x_1'+1)+x_1(1+y_1)+a_1'x_1']\}+(b_3-b_3')(1+2x_1)+2b_3x_1'\equiv0$ (mod 4).

(b) For $\lambda=1$, from (6), $b_2=-1-a_1+2^{m-8}b_3$, ($b_3=0\ldots7$; $a=1+2a_1$). (b)$\sim$the seven types $a=1$; $b_3=0$, $\beta_1=0$, 2; $a=3$, $b_3=\beta_1=0$, 1, 2; $m>8$, $a=1$; $m=8$, $a=3$; $b_3=1$, $\beta_1=0, 2$; with $C=\begin{bmatrix} P, & QP^{2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $b_3'-b_3+2ax_1'\equiv0$ (mod 8),* except for β_1 odd, where $C=\begin{bmatrix} P^{\beta_1'}, & QP^{2(\beta_1'-1)+2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$ where x_1' satisfies $b_3'\beta_1'-b_3+2ax_1'\equiv0$ (mod 8).

(B_3) $a=1, 3$; $m=7$. (B_3), ($a=3$)$\sim$(B_3), ($a=1$); with $C=\begin{bmatrix} P^3, & Q \\ P, & Q \end{bmatrix}$.

(a) $\lambda=0$. There are four types $b_2=1$, 3, $\beta_1=0$, 2.

(b) $\lambda=1$. (b)$\sim$the five types $b_2=0$, 1, $\beta_1=0$, 2, and $b_2=0$, $\beta_1=1$; with $C=\begin{bmatrix} P, & QP^{4x_1'} \\ P, & Q \end{bmatrix}$ where x_1' satisfies $b_2'-b_2+2ax_1'\equiv0$ (mod 4), except for β_1 odd, where $C=\begin{bmatrix} P^x, & Q^{y'}P^{2x_1'} \\ P, & Q \end{bmatrix}$, where x and y' are odd and the variables satisfy $x_1+x_1y_1'+x_1'y_1'-b_3'\equiv0$ (mod 2); $\beta_1'\equiv y'+2x_1'$ (mod 4); and $x_1+x_1'+y_1'\equiv0$ (mod 2).

Hence in (B) there are seventeen types for $m>8$, sixteen for $m=8$, ten for $m=7$, viz:

$Q^{-2}PQ^2=P^{1+2^{m-5}\beta_1}$, $P^{2^{m-3}}=1$.

$Q^{-1}PQ=Q^4P^{1+2^{m-6}b_2}$, $Q^8=1$; $m>8$; $b_2=\beta_1=0$, 1, 2; $m=8$, $b_2=\beta_1=0$, 2; $m=7$, $b_2=\beta_1=0$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-4}b_3}$, $Q^8=1$; $m>7$, $b_3=1$, $\beta_1=0$; $m=7$, $b_3=0$, 1, $\beta_1=0, 2$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-5}b_3}$, $Q^8=P^{2^{m-4}\lambda_1}$; $m>7$; $b_3=\beta_1=0$, 1, 2; $\lambda_1=0, 1$; $m=7$, $b_3=1$, $\beta_1=0$, 1, 2, $\lambda_1=1$.

$Q^{-1}PQ=Q^{2(1+2a_1)}P^{-1-4a_1+2^{m-6}b_3}$, $Q^8=P^8$; $a_1=b_3=0$, $\beta_1=0$, 2; $m>8$, $a_1=0$, $m=8$, $a_1=1$, $b_3=1$, $\beta_1=0$, 2; $m=7$, $a_1=1$, $b_3=\beta_3=0$, 1, 2.

(C) *$a=1$, $a=1$, 2, 3.*

$$Q^{-1}PQ=Q^{2a}P^b\ldots(1),$$

$$Q^{-2}PQ^2=Q^4P^\beta\ldots(2),$$

$$Q^{-4}PQ^4=P^{\omega+2^{m-4}\kappa}\ldots(3).$$

*For $m=8$, b_3 odd and $a=3$, x_1'satisfies $b_3'-b_3+2x_1'\equiv0$ (mod 8).

From (3), $[0,\ -4y_1,\ x,\ 0,\ 4y_1]=[0,\ 0,\ x(\omega+2^{m-4}\kappa)^{y_1}]\ldots(4)$, and

$$[0,\ 4y_1,\ x]^s=[0,\ 4sy_1,\ (s-\theta_s)\{\frac{1+(\omega+2^{m-4}\kappa)^{y_1}}{2}\}x+\theta_s x]\ldots(5).$$

Now $\omega=1$; for when $\omega=-1$, (2) raised to the second power shows that P is of an order lower than 2^{m-3}. Also a is odd; for if $a=2$, (1) transformed by Q and placed equal to (2) shows that b is even, which makes P of an order lower than 2^{m-3}. Next transform (2) by Q^2; $\beta=1+2\beta_1$. From (5) and (2),

$$[0,\ -2y_1,\ x,\ 0,\ 2y_1]=[0,\ 4x\theta_{y_1},\ x(1+2\beta_1\theta_{y_1})+2^{m-5}\kappa(xy_1-\theta_{xy_1})]\ldots(6).$$

Hence,

$$[0, 2y_1, x]^s=[0, 2sy_1+2(s-\theta_s)x\theta_{y_1}, sx+2(s-\theta_s)\{\beta_2 x\theta_{y_1}+2^{m-7}\kappa(xy_1-\theta_{xy_1})\}].(7).$$

Square (1) by (7); $b=1+2b_1$, $\beta_1=2\beta_2$. From (1) and (7),

$$[0,\ -y,\ x,\ 0,\ y]=[0,\ 4x\theta_{y_1}+2\theta_y\{ax+b(x-\theta_x)\},\ 16\lambda\beta_2 x\theta_{yy_1}(1+a_1+b_1)+\{x(1+4\beta_2\theta_{y_1})+2^{m-5}\kappa(xy_1-\theta_{xy_1})\}\{1+2\theta_y(b_1+\beta_2+2b_1\beta_2)\}+2\theta_y\{-\theta_x\beta_2(1+2b_1)+2^{m-6}\kappa(a_1+b_1)(x-\theta_x)\}]\ldots(8).$$

Hence,

$$[0,\ y,\ x]^2=[0,\ 4x\theta_{y_1}+2\{y+\theta_y[ax+b(x-\theta_x)]\},\ 16\lambda\beta_2\theta_{yy_1}x(1+a_1+b_1)+x+x(1+4\beta_2\theta_{y_1})\{1+2\theta_y(b_1+\beta_2+2\beta_2 b_1)\}+2^{m-5}\kappa(xy_1-\theta_{xy_1})+2\theta_y\{2^{m-6}\kappa(a_1+b_1)(x-\theta_x)-\beta_2\theta_x(1+2b_1)\}]\ldots(9).$$

Now transform (1) and (2) by Q^2 and Q, respectively; then raise them to the power 8λ. There result:

$\beta_2=-\lambda+2^{m-7}\beta_3$, $(\beta_0=0\ldots3)$; $\beta_3\equiv\kappa \pmod 2\ldots(10)$;

$\lambda\{\lambda(1+2a_1)+b_1\}\equiv0 \pmod{2^{m-7}}\ldots(11)$, and

$(1+b_1)[\lambda(1+2a_1)+b_1]+2^{m-7}[\beta_3(b_1-1)+2\kappa b_1 a_1]\equiv0 \pmod{2^{m-5}}\ldots(12)$.

(C_1) $b=2b_2$, $m>7$. From (11), $\lambda=2\lambda_1$. Let $Q'=QP^{2x_1}$; then $Q'^8=1$, where x_1 satisfies $\lambda_1(1+2ax_1)+x_1(1+2b_2)\equiv0 \pmod{2^{m-5}}$. From (12), $b_1=2^{m-7}b_3$ $(b_3=0\ldots7)$. Thus $(C_1)\sim$the four types $a=1,\ 3$; $\kappa=\beta_3=b_3=0,\ 1$; with $C=\begin{bmatrix}P, & QP^{2^{m-6}x_1'}\\ P, & Q\end{bmatrix}$, where x_1' satisfies $\beta_3-\beta_3'+2x_1'\equiv0 \pmod 4$, and $b_3-b_3'\equiv2a'x_1' \pmod 8$.

(C_2) $b=1+2b_2$, $m>7$.

(a) For λ even, from (12), $b_1=-1+2^{m-6}b_3$ $(b_3=0\ldots3)$. From (11), $\lambda=2^{m-7}\lambda_2$ $(\lambda_2=0,\ 1)$. Hence $(a)\sim$the fourteen types $\lambda_2=0,\ 1$; $\kappa=\beta_3=b_3=0,\ 1,\ a=1$; $\kappa=\beta_3=0,\ 1,\ b_3=0,\ a=3$; $\lambda_2=\kappa=0,\ \beta_3=0,\ 2,\ b_3=2,\ a=1,\ 3$; $\lambda_2=\kappa=1,\ \beta_3=3$; $b_3=a=1$; $b_3=0,\ a=3$. The isomorphism is given by $C=\begin{bmatrix}Q^{2y_1}P^x, & QP^{2x_1'}\\ P, & Q\end{bmatrix}$ where $x=1+2x_1$, and the variables satisfy $(\beta_3-\beta_3')x+2[\beta_3 x_1'+\lambda_2 x_1'+\kappa x_1]\equiv0 \pmod 4$, and $\lambda_2\{2a_1x_1+x_1+y_1-x_1'\}-\kappa\{2(a_1-1)(x_1+x_1')-(y_1-\theta_{y_1})\}-\beta_3\{x_1-a'x_1'-\theta_{y_1}\}-2b_3x_1'+(b_3-b_3')x\equiv0 \pmod 4$.

(b) For λ odd, from (11) $b_1=-1-2a_1+2^{m-7}b_3$. Let $Q'=Q^y$ (y odd). Then $Q'^8=P^8$ where y satisfies $\lambda_y\equiv 1 \pmod{2^{m-6}}$. Thus ($b$)$\sim$the seven types $\kappa=\beta_3=0$; $b_3=0$, 2, $a=1$, 3; $b_3=1$, $a=1$;* $\kappa=\beta_3=1$, $b_3=1$, 3; with $C=\begin{bmatrix} P, & QP^{2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3-\beta_3'+2x_1'\equiv 0 \pmod 4$, and $b_3-b_3'-2x_1'(1+2a_1)\equiv 0 \pmod 8$;† except for $a=1$, $\kappa=0$; $\beta_3=0$, $b_3=3+4b_4$;‡ $\beta_3=2$, $b_3=1+4b_4$, $b_4=0$, 1; where $C=\begin{bmatrix} Q^2P^x, & Q \\ P, & Q \end{bmatrix}$, with $x=1+2(2^{m-6}-1)$, -1, $1+2(\pm 2^{m-7}-1)$, respectively.

(C_3) $m=7$. From (10), $\beta_2=-\lambda-\kappa+2\beta_3$.

(a) $b_1=2b_2$. Here (a)$\sim$the four types, $\beta_3=0$; $a=1$, 3, $\kappa=b_2=\lambda=0$; $\kappa=\lambda=1$, $a=1$, $b_2=1$, $a=3$, $b_2=0$, with $C=\begin{bmatrix} P, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3+\beta_3'\equiv x_1' \pmod 2$ and $b_2-b_2'-x_1'(\lambda-\kappa+a+2\beta_3+2b_2)\equiv 0 \pmod 4$.

(b) $b_1=1+2b_2$. Here (b)$\sim$the fourteen types, $a=1$, 3, $\kappa=0$; $\lambda=0$, $\beta_3=0$, $b_2=1$, 3; $\beta_3=1$, $b_2=1$; $\lambda=1$, $\beta_3=0$, $b_3=0$, 1; $\kappa=1$, $\beta_3=0$; $b_2=0$; $a=3$, $\lambda=0$; $a=1$, $\lambda=1$; $b_2=1$, 3, $a=3$, $\lambda=1$; with $C=\begin{bmatrix} P^x, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where $x=1+2x_1$ and the variables satisfy $\beta_3+\beta_3'+x_1'(\lambda-\kappa)-\lambda x_1\equiv 0 \pmod 2$, and $(b_2-b_2')x+(\lambda-\kappa)(ax_1-x_1')-2\beta_3(x_1+x_1')-2x_1'(1+b_2)\equiv 0 \pmod 4$. Hence in ($C$) there are twenty-five types for $m>7$; eighteen for $m=7$. The defining relations of these types are the following:

$Q^{-4}PQ^4=P^{1+2^{m-4}\kappa}$, $P^{2^{m-3}}=1$.

$Q^{-1}PQ=Q^{2a}P^{\omega_1+2^{m-6}b_3}$, $Q^{-2}PQ^2=Q^4P^{1+2^{m-5}\beta_3}$, $Q^8=1$; $a=1$, 3; $\omega_1=1$, $m>7$, $b_3=\beta_3=\kappa=0$, 1; $m=7$, $b_3=\beta_3=\kappa=0$; $\omega_1=-1$, $b_3=4$, $\beta_3=0$, 2, $\kappa=0$; $m=7$, $b_3=\beta_3=\kappa=0$.

$Q^{-1}PQ=Q^{2a}P^{-1+2^{m-5}b_3}$, $Q^{-2}PQ^2=Q^4P^{1+2^{m-5}(\beta_3-\lambda_2)}$, $Q^8=P^{2^{m-4}\lambda_2}$; $m>7$; $\kappa=0$, $b_3=\beta_3=0$, $a=1$, 3, $\lambda_2=0$, 1; $\kappa=1$; $\beta_3=1$, $\lambda_2=0$, 1; $a=1$, $b_3=1$; $a=3$, $b_3=0$; $\beta_3=3$, $\lambda_2=1$, $a=1$, $b_3=0$, 1; $m=7$, $a=3$, $\kappa=\beta_3=1$, $\lambda=b_3=0$.

$Q^{-1}PQ=Q^{2a}P^{-1-4a_1+2^{m-6}b_3}$, $Q^{-2}PQ^2=Q^4P^{1-4-2^{m-5}\beta_3}$, $Q^8=P^8$; $\beta_3=\kappa=0$; $a=1$, 3, $b_3=0$, 2; $m>8$, $a=1$, $b_3=1$; $m=8$, $a=3$, $b_3=1$; $\beta_3=\kappa=1$; $m>7$, $a=3$, $b_3=1$, 3; $m=7$, $a=1$, $b_3=2$, 3; $a=3$, $b_3=0$, 2, 6.

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*For $m=8$, $b_3=1$, $a=3$.

†For $m=8$, $b_3-b_3'-2ax_1'[1+2a_1+2\beta_3+2b_3+2\kappa(a_1+1)]\equiv 0 \pmod 8$.

‡For $m=8$, $\beta_3=0$, $b_3=1+4b_4$; $\beta_3=2$, $b_3=3+4b_4$.

PART 2. Q^4 IS IN $\{P\}$ AND Q^2 IS NOT IN $\{P\}$.

We have $Q^4=P^{4\lambda}$. Two cases arise. They will be treated in separate sections.

§1. $\{P,\ Q\}$ IS OF ORDER 2^{m-1}.

Here $\{P,\ Q\}=G_{m-1}$ is determined from the types in III, Introduction, by replacing m by $m-1$. The types of G_{m-1} are now designated by (A), (B)... Thus writing $Q^{-1}PQ=P^{\omega+2^{m-5}\beta_1}$...(1), $Q^4=P^{2^{m-4}\lambda_1}$, we have

(A) and (B), $\omega=\pm 1$, $\beta_1=0,\ 1,\ 2$, $\lambda_1=0$; (C), $\omega=-1$, $\beta_1=0$, $\lambda_1=1$; and

$Q^{-2}PQ^2=Q^{1+2^{m-4}\kappa}$ ...(2);

(D), $Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2}$...(1), $Q^4=1$; $\beta_2=0,\ 1$, $\kappa=0,\ 1$;

(E), $Q^{-1}PQ=Q^2P^{1-2^{m-5}\kappa}$ ...(1), $Q^4=1$, $\kappa=0,\ 1$;

(F), $Q^{-1}PQ=Q^2P^{-1}$...(1), $Q^4=P^{2^{m-4}}$;

(G), $Q^{-1}PQ=Q^2P^{-1+2^{m-5}\beta_2}$...(1), $Q^4=P^4$, $\beta_2=0,\ 1$, $\kappa=0$; $\beta_2=0$, $\kappa=1$.

Let R be an operator of G_m, not in G_{m-1}. Since R^2 is in G_{m-1}, $R^2=Q^{\mu}P^{\nu}$...(3). Then $G_m=[R,\ G_{m-1}]$. In G_m, $R^{-1}PR=Q^aP^b$...(4), and $R^{-1}QR=Q^cP^d$...(5). Consider R with each G_{m-1}. For (A), (B), and (C), $[0,\ -y,\ x,\ 0,\ y]=[0,\ 0,\ x(\omega+2^{m-5}\beta_1)^y]$...(6). Hence $[0,\ y,\ x]^s=[0,\ sy,\ (s-\theta_s)x\{\frac{1+(\omega)^y}{2}+(\omega)^{y-1}\times 2^{m-6}\beta_1 y\}+\theta_s x\{1+2^{m-5}\beta_1(s-1)y\}$...(7).

(A) From (3), (4), and (5), by means of (7), $b=1+2b_1$, $d=2^{m-5}d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. $(RP)^4$ is in $\{P\}$. Hence $a=2a_1$. Transforming (1), (3), (4), and (5) by R, we get $c=1+2c_1$, and the congruences

$\beta_1c_1\equiv 0 \pmod 2$...(8), $2^{m-6}(d_1\mu_1+a_1\nu_1\beta_1)+b_1\nu_1\equiv 0 \pmod{2^{m-5}}$...(9),

$2^{m-6}a_1d_1+b_1(1+b_1)+2^{m-6}(a_1b_1\beta_1-\beta_1\mu_1)\equiv 0 \pmod{2^{m-5}}$...(10),

and $d_1(1+b_1+c_1)\equiv\beta_1\nu_1 \pmod 2$...(11).

$[z,\ y,\ x]^{2s_1}=[0,\ 2\{y+z(\mu_1+c_1y+a_1x)\}s_1,\ 2s_1\{x+2^{m-6}\beta_1xy+z(\nu_1+b_1x+2^{m-5}b_1\beta_1xy+2^{m-6}d_1y+2^{m-6}a_1\beta_1(x-\theta_x)]\}]$...(12), and

$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1z(\mu_1+c_1y+a_1x),\ x(1+2s_1)+2^{m-4}\beta_1s_1x\{y+z(\mu_1+c_1y+a_1x)\}+2s_1\{2^{m-6}\beta_1xy+z[\nu_1+b_1x+2^{m-5}b_1\beta_1xy+2^{m-6}d_1y+2^{m-6}a_1\beta_1(x-\theta_x)]\}]$...(13).

By means of the general correspondence $\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where $P'=R^zQ^yP^x$, $Q'=R^{z'}Q^{y'}P^{x'}$, $R'=R^{z''}Q^{y''}P^{x''}$, the groups in (A) are simply isomorphic with the types given below.

(A_1) $b_1=2^{m-6}b_2$ $(b_2=0...3)$. x is odd, $x'=2^{m-6}x_1'$, $x''=2^{m-6}x_1''$, and the variables satisfy the congruences $y'+z'(\mu_1+c_1y')=k$ (k odd), $x_1'+d_1y'z'\equiv 0$

(mod 2), $x_1''+d_1y''z''\equiv 0$ (mod 2), $a_1z'+c_1(yz'-y'z)\equiv 0$ (mod 2), $b_2(xz'-2^{m-6}x_1'z)+\beta_1(xy'-2^{m-6}x_1'y)+2a_1\beta_1(x_1z'-2^{m-7}x_1'z)+d_1(yz'-y'z)\equiv\beta_1'\{x+2^{m-6}(\beta_1xy+b_2xz+d_1yz)\}$ (mod 4), $a_1z''+c_1(yz''-y''z)\equiv a_1'[y'+z'(\mu_1+c_1y')]$ (mod 2), $b_2(xz''-2^{m-6}x_1''z)+\beta_1(xy''-2^{m-6}x_1''y)+2a_1\beta_1[2x_1z''-2^{m-6}x_1''z]+d_1(yz''-y''z)\equiv 2d_1a_1'z[y'+z'(\mu_1+c_1y')]+a_1'[2^{m-6}\beta_1x_1'y'+z'(d_1y'+2^{m-6}a_1\beta_1x_1')]+a_1'x_1'\times(1+2^{m-6}b_2z')+b_2'[x(1+2^{m-6}\beta_1y)+2^{m-6}z(b_2x+d_1y)]$ (mod 4), $c_1(y'z''-y''z')\equiv c_1'[y'+z'(\mu_1+c_1y')]$(mod2), $2^{m-6}\{b_2(x_1'z''-x_1''z')+\beta_1[x_1'y''-x_1''y'+a_1(x_1'z''-x_1''z')]\}+d_1(y'z''-y''z')\equiv c_1'\{x_1'[1+2^{m-6}(b_2z'+\beta_1y'+a_1\beta_1z')]+d_1y'z'\}+d_1'[x+2^{m-6}(\beta_1xy+b_2xz+d_1yz)]$ (mod 4), $y''+z''(\mu_1+c_1y'')\equiv\mu_1'[y'+z'(\mu_1+c_1y')]$ (mod 2), $x_1''+2^{m-6}\beta_1x_1''y''+z''[d_1y''+2^{m-6}x_1''(b_2+a_1\beta_1)]\equiv\mu_1'\{x_1'+2^{m-6}\beta_1x_1'y'+z'[d_1y'+2^{m-6}x_1'(b_2+a_1\beta_1)]\}$ (mod 4).

(A_2) $b_1=-1+2^{m-6}b_2$, $(b_2=0\ldots 3)$. In this case x and y' are odd, $z=z'=0$, $z''=1$, $x'=2^{m-5}x_2'$, and the variables satisfy the congruences $b_2x''+\beta_1x''y''++d_1y''\equiv 0$ (mod 2), $\beta_1y'\equiv\beta_1'(1+2^{m-6}\beta_1y)$(mod 4), $a_1'\equiv a_1+y(1+c_1)$ (mod 2), $b_2x-\beta_1\{xy''+x''y-2a_1(x_1+x'')-2c_1x''y+xy\}+d_1y\equiv 2a_1'x_2'+b_2'x(1+2^{m-6}\beta_1y)$ (mod 4), $c_1\equiv c_1'$ (mod 2), $2x_2'(1+c_1)-\beta_1x''(y'-2c_1)+d_1y'\equiv d_1'x(1+2^{m-6}\beta_1y)$ (mod 4), $\mu_1+y''(1+c_1)+a_1x''\equiv\mu_1'$ (mod2), $b_2x''+d_1y''+\beta_1[a_1(x''-\theta_{x''})-x''y'']+2\nu_3\equiv 2\mu_1'x_2'+\nu_2'x(1+2^{m-6}\beta_1y)$ (mod 4). There are eighteen types in (A_1), twenty-eight in (A_2). These are tabulated below.

$$Q^{-1}PQ=P^{1+2^{m-5}\beta_1},\ R^{-1}PR=Q^{2a_1}P^{\omega_1+2^{m-5}b_2},\ R^{-1}QR=Q^{1+2c_1}P^{2^{m-5}d_1},$$
$$R^2=Q^{2\mu_1}P^{2^{m-4}\nu_2},\ Q^4=1,\ P^{2^{m-3}}=1.$$

a_1	c_1	μ_1	ν_2	b_2	d_1	β_1	ω_1	a_1	c_1	μ_1	ν_2	b_2	d_1	β_1	a_1	c_1	μ_1	ν_2	b_2	d_1	β_1
0,1	0	0	0	0	0	0,1,2	1	0	1	0	0	2	2	2	0	1	1	0	0	0	0,1
0	0,1	0	0	2	0	0	1	0	0	0	1	0	0	0,2	0,1	1	1	0	2	0	0
0	1	0	0	0	0,1	0,2	1	1	1	0	1	0	0	0,2	0	1	0	0	0	2	0
0	1	1	0	0,2	0	0	1	0	1	0	1	0	0	0	0	1	0	0	2	0	0
0,1	0	0	0	0	2	0,1	1	0,1	1	0	0	0	0	0,2	0	0	0	0	2	1	1
0	0	0	0	0	0,1	0,2	−1	1	1	0	0	0	2	0,2	0	1	1	0	0	2	0
0	0	0	0	2	0	0,2	−1	0	0	0	0	0	0	1	1	1	1	0	2	2	2
										$\omega_1=-1$							$\omega_1=-1$				

(B) From (3) and (4), by means of (7), $a=2a_1$, $b=1+2b_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. Also $(RQ)^4$ is in $\{P\}$; hence $c=1+2c_1$, and $d=2d_1$. Transforming (1), (3), (4), and (5) by R, we obtain

$\beta_1(a_1+c_1)\equiv 0$ (mod 2)...(8),

$\nu_1b_1+2^{m-6}\beta_1(\mu_1d_1+a_1\nu_1)\equiv 0$ (mod 2^{m-5})...(9),

$d_1(1+b_1)-\nu_1(1-2^{m-6}\beta_1)\equiv 0$ (mod 2^{m-5})...(10), and

$b_1(1+b_1)+2^{m-6}\beta_1[a_1(b_1+d_1)+\mu_1]\equiv 0$ (mod 2^{m-5})...(11).

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{(\phi_y+(-1)^{y-1}2^{m-6}\beta_1 y)x+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\beta_1 y)+2^{m-6}\beta_1(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$.(12),

and $[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1z(\mu_1+c_1y+a_1x),\ x\{1-2^{m-4}\beta_1\ s_1[y+z(\mu_1+c_1y+a_1x))\}+2s_1\{x(\phi_y+(-1)^{y-1}2^{m-6}\beta_1y)+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\beta_1y)+2^{m-6}\beta_1(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(13).

The groups in (B) for ν_1 other than 0 are isomorphic with those for $\nu_1=0$; with $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $\nu_1+x''\theta_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})(1-2^{m-5}\beta_1y'')+2^{m-6}\beta_1[a_1(x''-\theta_{x''})+d_1(y''-\theta_{y''})-(-1)^{y''}x''y'']\equiv\nu_1'$ (mod 2^{m-4}).

(B_1) $b_1=2^{m-6}b_2$. From (10), $d_1=2^{m-5}d_3$.

(B_2) $b_1=-1+2^{m-6}b_2$. All the groups in (B_2) are isomorphic with those where $d_1=0,\ 1,\ 2$; with $C=\begin{bmatrix} P, & QP^{x}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x' and x'' satisfy $\beta_1x'\equiv0$ (mod 2), $\mu_1+a_1x''\equiv\mu_1'$ (mod 2), $b_2x''+\beta_1a_1(x''-\theta_{x''})\equiv\beta_1\mu_1'x'$ (mod 4), and $x''+b_1x'+d_1-2^{m-6}\beta_1[x''(1+2c_1+2a_1x')+a_1(x'-\theta_{x'})+3c_1'x']\equiv d_1'$ (mod 2^{m-4}).

The groups in (B) are simply isomorphic with the types given below, or with (A). The correspondence is given by $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where x and y' are odd. For (B_1), y is even. For (B_2), $y+z\equiv0$ (mod 2); except for the groups $\sim(A)$, where for (B_1) x and y'' are odd, y and y' even, $z'=1$, $x'=2^{m-5}x_1'$; and for (B_2) $P'=P$, $Q'=RQP^{x'}$, $R'=R$. The variables satisfy the congruences derived in the usual way from the isomorphism. These congruences are somewhat complicated, but similar to those derived in previous cases. They are omitted for the sake of brevity.

There are in (B_1) twenty-two types; in (B_2) two for $m>7$, one for $m=7$, viz:

(B_1) $Q^{-1}PQ=P^{-1+2^{m-5}\beta_1}$, $R^{-1}PR=Q^{2a_1}P^{1+2^{m-5}b_2}$, $R^{-1}QR=Q^{1+2c_1}P^{2^{m-4}d_1}$, $R^2=Q^{2\mu_1}$, $P^{2^{m-3}}=1$.

a_1	c_1	μ_1	d_3	b_2	β_1	a_1	c_1	μ_1	d_3	b_2	β_1	a_1	c_1	ν_1	d_3	b_2	β_1
0	0	0	0	0	0,1,2	0	0	0	0	2	0	0	0	0	1	0	0,1
0	1	0	0	0,2	0,2	0	1	0	1	0	0	1	0	0	0	0,2	0,2
1	0	0	1	0,2	0	1	1	0	0,1	0	1	1	0	1	1	0,2	0

$b_2=2,\ \beta_1=2.$

(B_2) $Q^{-1}PQ=P^{-1+2^{m-5}}$, $R^{-1}PR=Q^2P^{-1+2^{m-4}b_2}$, $R^{-1}QR=Q^3P^2$, $R^2=Q^4$, $P^{2^{m-3}}=1$; $m>7$, $b_2=0,\ 1$; $m=7$, $b_2=0$.

(C) From (3) and (4) by means of (7), $a=2a_1$, $b=1+2b_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. The transformation of (1) by R shows that $c=1+2c_1$, $a_1=0$. Also RQ must be of order 2^{m-3} at most; hence $d=2d_1$. Transforming (3), (4), and (5) by R, we obtain

$b_1\nu_1+2^{m-6}c_1\mu_1\equiv 0 \pmod{2^{m-5}}$...(8),

$b_1(1+b_1)\equiv 0 \pmod{2^{m-5}}$...(9),

$d_1(1+b_1)\equiv \nu_1 \pmod{2^{m-5}}$...(10),

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y+z(\mu_1+c_1y)\},\ 2s_1\{\phi_y x+(-1)^y z(d_1\theta_y+b_1x)+\nu_1 z\}]$...(11),

and $[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1z(\mu_1+c_1y), x+2s_1\{\phi_y x+(-1)^y z(d_1\theta_y+b_1x)$
$+\nu_1 z\}]$...(12).

If $Q'^4=(RQ^{y'}P^{x'})^4=1$, and $Q'^2\neq\{P\}$, then $\mu_1+y'(1+c_1)$ is odd (13), and $x'\phi_{y'}+(-1)^{y'}(d_1\theta_{y'}+b_1x')+2^{m-6}[\mu_1+y'(1+c_1)]\equiv 0 \pmod{2^{m-5}}$. The groups satisfying these conditions do not belong in (C).

The groups for ν_1 other than 0 are isomorphic with those for $\nu_1=0$, through $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $2^{m-5}[\mu_1+\mu_1'+y''(1+c_1)]+x''\phi_{y''}+(-1)^{y''}[d_1\theta_{y''}+b_1x'']\equiv \nu_1'-\nu_1 \pmod{2^{m-4}}$.

(C_1) $b_1=2^{m-5}b_2$. From (10), $d_1=2^{m-5}d_2$. (C_1)$\sim$the four types given below, with $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where x and y' are odd, y and y'' even, $z'=z=0$, $z''=1$, $x''=2^{m-5}x''$, and the variables satisfy $y_1\equiv 0 \pmod 2$, $d_2+d_2'+b_2x'+x_1''\equiv 0 \pmod 2$, $y_1''(1+c_1)-\mu_1y_1'\equiv x_1'' \pmod 2$.

(C_2) $b_1=-1+2^{m-5}b_2$. All the groups in (C_2) are isomorphic with those where $d_1=0$, by $C=\begin{bmatrix} P, & QP^{x'}, & RP^{2x_1''} \\ P, & Q, & R, \end{bmatrix}$, where x' and x_1'' satisfy $d_1'\equiv d_1+b_1x'+2x_1'' \pmod{2^{m-4}}$. Also ($C_2$)$\sim$($C_1$), with $C=\begin{bmatrix} P, & Q, & RQP^{2^{m-5}} \\ P, & Q, & R \end{bmatrix}$.

In (C) there are the following four types:

$$Q^{-1}PQ=P^{-1},\ R^{-1}PR=P^{1+2^{m-4}b_2}, R^{-1}QR=Q^{1+2c_1},\ R^2=1,\ Q^4=P^{2^{m-4}}, P^{2^{m-3}}=1,$$
$$c_1=0,\ 1,\ b_2=0,\ 1.$$

For (D), (E), (F), and (G),

$[0,\ -2y_1,\ x,\ 0,\ 2y_1]=[0,\ 0,\ x(1+2^{m-4}\kappa y_1)]$...(6), and

$[0,\ 2y_1,\ x]^s=[0,\ 2sy_1,\ sx+2^{m-5}\kappa xy_1(s-\theta_s)]$...(7).

(D) From (1), $[0,\ -y,\ x,\ 0,\ y]=[0,\ 2xy,\ (-1)^y x+2^{m-5}\kappa\{x(y-\theta_y)+y(x-\theta_x)\}+2^{m-4}\beta_2xy]$...(8). Hence

$[0,\ y,\ x]^s=[0,\ sy+(s-\theta_s)xy,\ \theta_s x+(s-\theta_s)\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\alpha_x)]+$
$2^{m-5}\beta_2xy\}]$...(9).

From (3) and (4) by means of (9), $\mu+\nu\equiv 0 \pmod 2$, $a=2a_1$, $b=1+2b_1$. If $c=2c_1$, or if c and d were both odd, Q^2, transformed by R^2 would result in $P^{2N}=Q^2P^{2^{m-4}M}$, which is impossible. Hence $c=1+2c_1$, and $d=2d_1$. Transforming (1), (3), (4), and (5) by R, we get $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$\kappa(a_1+b_1+c_1+d_1)\equiv 0 \pmod 2$...(10),

$\nu_1 b_1+2^{m-6}\kappa(\mu_1 d_1+\nu_1 a_1)\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}\kappa[a_1(b_1+d_1)+\mu_1]\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+b_1)-\nu_1+2^{m-6}\kappa[d_1(a_1+c_1)+\nu_1]\equiv 0 \pmod{2^{m-3}}$...(13),

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y(1+x)+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+ y(x-\theta_x)]+2^{m-5}\beta_2xy+z[\nu_1+(b_1x+d_1\theta_y)((-1)^y+2^{m-5}\kappa y)+2^{m-6}\kappa(a_1(x- \theta_x)+d_1(y-\theta_y))]\}]$...(14), and

$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1\{xy+z(\mu_1+c_1y+a_1x)\},\ x\{1+2^{m-4}\kappa s_1 z(\mu_1+c_1y +a_1x)\}+2s_1\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\theta_x)]+2^{m-5}\beta_2xy+z[\nu_1+(b_1x+ d_1\theta_y)((-1)^y+2^{m-5}\kappa y)+2^{m-6}\kappa(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(15).

The groups for ν_1 other than 0 are isomorphic with those for $\nu_1=0$ and $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, odd for b_1 odd, and x'' and y'' satisfy the congruence $\nu_1+x''\phi_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})+2^{m-6}\{\kappa[(x''+ d_1)(y''-\theta_{y''})+(y''+a_1)(x''-\theta_{x''})+2y''(b_1x''+d_1\theta_{y''})]+2\beta_2x''y''\}\equiv\nu_1' \pmod{2^{m-4}}$. For $\mu_1=1$, the groups where $a_1=0$ for b_1 even, and $a_1=1$ for b_1 odd, are isomorphic to (A), (B), or (C), with $C=\begin{bmatrix} P, & R, & Q^{y''}P^{x''} \\ P, & Q, & R \end{bmatrix}$, where y'' and x'' are odd and satisfy $\kappa(y_1''+x_1'')+\beta_2\equiv 0 \pmod 2$. Also the groups for $a_1=0$, b_1 odd are isomorphic to (A), (B), or (C), with $C=\begin{bmatrix} P, & R, & QP \\ P, & Q, & R \end{bmatrix}$.

(D_1) $b_1=2^{m-6}b_2$. From (13), $d_1=2^{m-5}d_2$.

(D_2) $b_1=-1+2^{m-5}b_3$. All the groups in (D_2)$\sim$those where $d_1= 0, 1$, with $C=\begin{bmatrix} P, & QP^{2x_1}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x_1' and x'' satisfy $\mu_1+a_1x''\equiv\mu_1' \pmod 2$, $b_2x''+2a_1\kappa x_1''\equiv 2\kappa\mu_1'x_1' \pmod 4$, $x''-2x_1'+d_1-d_1'+2^{m-5}\{x''(\beta_2+\kappa c_1)+ x_1'(b_2+\kappa+\kappa c_1')\}+2^{m-6}\kappa(x''+\theta_{x''})\equiv 0 \pmod{2^{m-4}}$, $c_1-x''\equiv c_1' \pmod 2$.

The groups in (D) are simply isomorphic with the types given below, or with those in the preceding cases through $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, x and y' are odd, x' even, $z'=0$, $z''=1$. For (D_1), y and y'' are even, and $x''=2^{m-5}x_1''$, and for (D_2), y'' is odd; except for the groups$\sim$(A), (B), or (C), where for (D_1') x is odd and y even, and for (D_2), $P'=P$, $Q'=RQP^{x'}$, $R'=R$. It has been verified that the variables, limited as above, satisfy the congruences derived in the usual way from C by means of the transforming relations of the group.

There are seventeen types in (D_1), and five in (D_2). They follow:

$$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=Q^{2a_1}P^{\omega_1+2^{m-4}b_2},$$
$$R^{-1}QR=Q^{1+2c_2}P^{2^{m-4}d_1},\quad R^2=Q^{2\mu_1},\quad Q^4=1,\quad P^{2^{m-3}}=1.$$

a_1	c_1	μ_1	d_2	b_2	κ	β_2	ω_1
0	0	0	0	0	0	0,1	1
1	1	0	0	0	0	0,1	
0	0	0	1	1	0	0,1	
1	1	0	1	1	0	0,1	

a_1	c_1	μ_1	d_2	b_2	κ	β_2	ω_1
0	0	0	0	1	0	0	1
1	1	0	0	1	0	0	
0	0	0	1	0	0	0	
1	1	0	1	0	0	0	

a_1	c_1	μ_1	d_2	b_2	κ	β_1	ω_1
0	0	0	0,1	0	1	0	1
1	1	0	0,1	0	1	0	
1	0	1	1	0	0	0	
0	0	0	0	0	0	0	−1
0	0	0	0	1	0	0,1	−1

$$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2},\quad Q^{-2}PQ^2=P,\quad R^{-1}PR=P^{-1+2^{m-4}},\quad R^{-1}QR=Q^3P^2,$$
$$R^2=1,\quad Q^4=1,\quad P^{2^{m-3}}=1,\quad \beta_2=0,\ 1.$$

(E) From (1), $[0,\ -y,\ x,\ 0,\ y]=[0,\ 2xy,\ x+2^{m-5}\kappa\{x(y-\theta_y)-\theta_{xy}\}]$...(8), and $[0,\ y,\ x]^s=[0,\ sy+(s-\theta_s)xy,\ sx+2^{m-6}(s-\theta_s)\kappa\{x(y-\theta_y)-\theta_{xy}\}]$...(9). From (3), (4), and (5), $\mu=2\mu_1$, $\nu=2\nu_1$, $b=1+2b_1$, $d=2^{m-5}d_1$. The operation $(RP)^4$ is in $\{P\}$, so $a=2a_1$. Transformation of (1), (3), (4), and (5) by R gives $c=1+2c_1$, and the congruences

$d_1+\kappa(b_1+c_1)\equiv 0 \pmod 2$...(10),

$b_1\nu_1+2^{m-6}(d_1\mu_1+\kappa\nu_1a_1)\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}[a_1(d_1+\kappa b_1)+\kappa\mu_1]=0 \pmod{2^{m-5}}$...(12), and

$d_1(1+b_1+c_1)\equiv 0 \pmod 2$...(13). Also,

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y(1+x)+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x+2^{m-6}\kappa[x(y-\theta_y)-\theta_{yx}]+z[\nu_1+b_1x+2^{m-6}\kappa a_1(x-\theta_x)+2^{m-6}d_1y]\}]$...(14),

$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1\{xy+z(\mu_1+c_1y+a_1x)\},\ x(1+2s_1)+2s_1\{2^{m-6}\kappa\times[x(y-\theta_y)-\theta_{yx}]+z[\nu_1+b_1x+2^{m-6}\kappa(a_1(x-\theta_x)+2(\mu_1+c_1y+a_1x))+2^{m-6}d_1y]\}]$...(15).

(E_1) $b_1=2^{m-6}b_2$. (E_2) $b_2=-1+2^{m-6}b_2$. From (11), $\nu_1=2^{m-6}\nu_2$. The groups in (E_1) and for b_2 odd, in (E_2), for ν_1 other than 0 are isomorphic with those for $\nu_1=0$. When b_2 is even in (E_2), and $\nu_2=3$, the groups are isomorphic with those for $\nu_2=1$. $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$, where $z''=1$, and x'' and y'' satisfy $\nu_1+x''(1+b_1)+2^{m-6}\kappa[a_1(x''-\theta_{x''})+x''(y''-\theta_{y''})-\theta_{x''y''}]+2^{m-6}d_1y''\equiv\nu_1' \pmod{2^{m-4}}$. For $a_1=0$, $\mu_1=1$, $\nu_1=0$, 2, $(E)\sim(A)$, (B) or (C); and for $\nu_1=0$, $C_1=\begin{bmatrix}P, & R, & Q\\ P, & Q, & R\end{bmatrix}$; and for $\nu_1=2$, $C=\begin{bmatrix}RP, & Q, & R\\ P & Q, & R\end{bmatrix}$. For $a_1=1$, and $\mu_1+c_1\equiv 0 \pmod 2$, $(E)\sim(A)$, (B) or (C) with $C=\begin{bmatrix}P, & RQP^{x'}, & R\\ P, & Q, & R\end{bmatrix}$, where x' satisfies $x'(1+b_1)+2^{m-6}(\kappa x'+d_1)\equiv 0 \pmod{2^{m-5}}$. For $a_1=0$, 1, $\mu_1=0$, $c_1=1$, (E_1), and for

$\kappa=0$, $(E_2)\sim(A)$, (B), or (C). For (E_1), $C=\begin{bmatrix} RP, & Q, & R \\ P, & Q, & R \end{bmatrix}$; for (E_2), $a_1=0$, $C=\begin{bmatrix} RP, & QP^{2^{m-5}x_1}, & R \\ P, & Q, & R \end{bmatrix}$, where $x_1'\equiv d_2 \pmod 2$; for (E_2), $a_1=1$, $C=\begin{bmatrix} QP, & RP, & R \\ P, & Q, & R \end{bmatrix}$. For $a_1=0$, $\mu_1+c_1\equiv 0 \pmod 2$, $\kappa=0$, $(E_2)\sim(D)$ with $C=\begin{bmatrix} P, & RQ, & R \\ P, & Q, & R \end{bmatrix}$. For $\kappa=d_1=0$ in (E_1) and for $\kappa=1$ in (E), for $a_1=\mu_1=c_1=0$, the groups where $b_2=2\sim$those where $b_2=0$; with $C=\begin{bmatrix} QP, & Q, & R \\ P, & Q, & R \end{bmatrix}$, for (E_1), $\kappa=0$, and for (E_2), $\kappa=1$; and with $C=\begin{bmatrix} P^3, & Q, & RQ^2 \\ P, & Q, & R \end{bmatrix}$, for (E_1), $\kappa=1$. In (E_2), for $\kappa=1$, the groups where $a_1=0$, 1, $\mu_1=0$, $c_1=1\sim$those where $a_1=c_1=\mu_1=0$, by $C=\begin{bmatrix} P, & Q, & RQ^{y''}P \\ P, & Q, & R \end{bmatrix}$, where for $a_1=0$, $y''\equiv 2b_3' \pmod 4$, and for $a_1=1$, $y''=1+2y_1''$, and y'' satisfies $b_3+d_2+y_1''\equiv 0 \pmod 2$, and $b_3+b_3'+y_1''\equiv 0 \pmod 2$, $(b_2=1+2b_3,\ b_2'=2b_3')$. Hence there are five types in (E_1), and two in (E_2), viz:

$$Q^{-1}PQ=Q^2P^{1-2^{m-5}\kappa},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=P^{\omega_1+2^{m-4}b_2},\quad R^{-1}QR=QP^{2^{m-5}d_1},\quad R^2=1,\quad Q^4=1,\quad P^{2^{m-3}}=1.$$

$\omega_1=1$; $\kappa=0$, 1, $d_1=0$, 1, $b_2=0$; $\kappa=0$, $d_1=0$, $b_2=1$; $\omega_1=-1$, $\kappa=1$, $b_2=0$, $d_1=1$, 3.

(F) From (1), $[0,\ -y,\ x,\ 0,\ y]=[1,\ 2x\theta_y,\ (-1)^y x-2^{m-5}\kappa(xy-\theta_{xy})]$...(8), and $[0,\ y,\ x]^s=[0,\ sy+(s-\theta_s)x\theta_y,\ \theta_s x+(s-\theta_s)\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})\}]$...(9).

From (3) and (4), by means of (9), $\mu+\nu\equiv 0 \pmod 2$, $a=2a_1$, $b=1+2b_1$. Transformation of (1), (3), (4), and (5) by R gives $c=1+2c_1$, $d=2d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$(1+\kappa)(a_1+b_1+c_1+d_1)\equiv 0 \pmod 2$...(10),

$b_1\nu_1+2^{m-6}\{\mu_1c_1+(d_1\mu_1+a_1\nu_1)(1+\kappa)\}\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}\{a_1(1+b_1+c_1+d_1)+\kappa(d_1+\mu_1+a_1b_1)\}\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+b_1)-\nu_1+2^{m-6}\{\nu_1+d_1(a_1+c_1)\}(1+\kappa)\equiv 0 \pmod{2^{m-5}}$...(13),

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y+x\theta_y+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\kappa y)+2^{m-6}c_1\kappa(x-\theta_x)+2^{m-6}d_1(1+\kappa)(y-\theta_y)]\}]$ ...(14), and

$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1\{x\theta_y+z(\mu_1+c_1y+a_1x)\},\ x\{1+2^{m-4}\kappa s_1[y+x\theta_y+z(\mu_1+c_1y+a_1x)]\}+2s_1\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\kappa y)+2^{m-6}a_1\kappa(x-\theta_x)+2^{m-6}d_1(1+\kappa)(y-\theta_y)]\}]$...(15).

If $Q'^4=(RQ^{y'}P^{x'})^4=1$, and $\{Q'^2\}\neq\{P\}$, then

$\mu_1+y'(1+c_1)+a_1x'+x'\theta_{y'}$ is odd...(16), and

$\nu_1+x'\phi_{y'}+(-1)^{y'}(b_1x'+d_1\theta_{y'})-2^{m-6}\{\kappa[y'(x'+2b_1x'+2d_1\theta_{y'})-\theta_{x'y'}]-\kappa a_1(x'-\theta_{x'})-d_1(1+\kappa)(y'-\theta_{y'})\}=2^{m-6}k$ (k odd)...(17).

The groups satisfying (16) and (17) do not belong in (F).

The groups for ν other than 0, are isomorphic with those for $\nu_1=1$, with $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $2^{m-6}\mu_1'+\nu_1'\equiv 2^{m-6}[\mu_1+y''(1+c_1)+a_1x''+x''\theta_{y''}]+\nu_1+x''\phi_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})(1-2^{m-5}\kappa)-2^{m-6}\{\kappa[x''y''-\theta_{x''y''}+a_1(x''-\theta_{x''})]+d_1(1+\kappa)(y''-\theta_{y''})\}$ (mod 2^{m-4}). For $\mu_1=1$, $a_1=0$, and for $a_1=1$, $\mu_1+c_1\equiv 0$ (mod 2), the groups are isomorphic with (A), (B), or (C), and for $a_1=0$, $C=\begin{bmatrix} P, & R, & Q \\ P, & Q, & R \end{bmatrix}$, and for $a_1=1$, $C=\begin{bmatrix} P, & RQP^{x'}, & R \\ P, & Q, & R \end{bmatrix}$, where x' satisfies $b_1x'+d_1\equiv 0$ (mod 2^{m-5}).

(F_1) $b_1=2^{m-6}b_2$. From (13), $d_1=2^{m-5}d_2$.

(F_2) $b_1=-1+2^{m-6}b_2$. All the groups in $(F_2)\sim$those where $d_1=0, 1, 2$ with $C=\begin{bmatrix} P, & QP^{2x_1'}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x_1' and x'' satisfy the congruences $2\mu_1x_1'(1+\kappa)-x''(a_1+b_2)+\kappa a_1(x''-\theta_{x''})\equiv 0$ (mod 4), $d_1'-d_1+2x_1'-x''+2^{m-6}\{c_1'[1+2x_1'(1+\kappa)]-c_1(1+2\kappa x'')+\kappa(x''+\theta_{x''})+2x_1'[a_1(1+\kappa)+b_2]+x''\}$ (mod 2^{m-4}), $c_1-x''\equiv c_1'$ (mod 2).

The groups in (F) are simply isomorphic with the types given below, or with those in preceding cases; where $Q^4=P^{2^{m-4}}$, and $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$. The variables x and y' are odd, x' and y even, $z'=0$, $z''=1$. For (F_1) y'' is even, and $x''=2^{m-5}x_1''$; for (F_2), $z=0$, and y'' is odd; except for the groups$\sim(C)$, where x, y' are odd, x' even, $z=1$, $z'=0$, and for (F_1) y is even, for (F_2) y is odd. The variables specified have been proven to satisfy the congruence conditions derived from the relations of the group.

There are eight types in (F_1), and none in (F_2). The groups of (F_1) are given by

$$Q^{-1}PQ=Q^2P^{-1},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=Q^{2a_1}P^{1+2^{m-4}b_2},\quad R^{-1}QR=Q^{1+2c_1}P^{2^{m-4}d_2},\quad R^2=1,\quad Q^4=P^{2^{m-4}},\quad P^{2^{m-3}}=1.$$

$\kappa=0, 1$, $b_2=0, 1$, $a_1=c_1=d_2=0$; $\kappa=0$, $b_2=0, 1$, $d_2=0, 1$, $a_1=c_1=1$.

(G) From (1),

$[0, -y, x, 0, y]=[0, 2x\theta_y, (-1)^yx+2^{m-5}\{\kappa(xy-\theta_{xy})+\beta_2x\theta_x\}]$...(8), and

$[0, y, x]^s=[0, sy+(s-\theta_s)x\theta_y, \theta_sx+(s-\theta_s)\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2x\theta_y]\}]$...(9).

From (3) and (4), by means of (9), $\mu+\nu\equiv 0$ (mod 2), and either $a=2a_1$, $b=1+2b_1$, or $a=1+2a_1$, $b=2b_1$. For the latter, $R^{-1}PR=Q^{1+2a_1}P^{2b_1}$, from which Q may be obtained in terms of R and P. Therefore the group is generated by R and P alone, and equals $\{R, P\}$, which belongs in §2 as well as in §1.

(G_{I}) $a=2a_1$, $b=1+2b_1$. Transforming (1), (3), (4), and (5), and $Q^4=P^4$ by R, we get $c=1+2c_1$, $d=2d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$a_1+b_1-c_1-d_1+2^{m-6}\beta_2(a_1+d_1)\equiv 0 \pmod{2^{m-5}}$...(10),

$\mu_1(c_1+d_1)+\nu_1(a_1+b_1)+2^{m-6}[\kappa(\mu_1 d_1+\nu_1 a_1)+\beta_2\mu_1 d_1]\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+a_1(1+b_1+c_1+d_1)+2^{m-6}[\kappa(a_1b_1+a_1d_1+\mu_1)+\beta_2 a_1 d_1]\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+a_1+b_1+c_1)+c_1(1+c_1)+2^{m-6}[\kappa(a_1d_1+c_1d_1+\nu_1)+\beta_2(c_1d_1+\nu_1)]\equiv 0 \pmod{2^{m-5}}$...(13),

$a_1+b_1-c_1-d_1\equiv 0 \pmod{2^{m-6}}$...(14),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+x\theta_y+z(\mu_1+c_1y+a_1x)\}, 2s_1\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2 x\theta_y] +z[\nu_1+(b_1x+d_1y)(1+2^{m-5}y(\kappa+\beta_2))+2^{m-6}(\kappa a_1(x-\theta_x)+d_1(\kappa+\beta_2)\times(y-\theta_y))]\}]$...(15), and

$[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1\{x\theta_y+z(\mu_1+c_1y+a_1x)\}, x[1+2^{m-4}\kappa s_1 z(\mu_1+c_1y+a_1x)]+2s_1\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2 x\theta_y]+z[\nu_1+(b_2x+d_1y)(1+2^{m-5}y(\kappa+\beta_2))+2^{m-6}(\kappa a_1(x-\theta_x)+d_1(\kappa+\beta_2)(y-\theta_y))]\}]$...(16).

If $Q'^{2^{m-3}}=(RQ^{y'}P^{x'})^{2^{m-3}}\neq 1$, and $Q'^2\neq\{P\}$, then $\mu_1+y'(1+c_1)+x'(a_1+\theta y')=k$...(17), and $x'\phi_{y'}+b_1x'+d_1y'=k'$...(18) ($k$ and k' odd). The groups satisfying (17) and (18) do not belong to (G).

The groups for ν_1 other than $0\sim$ those for $\nu_1=0$, with $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$ where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd; and x'' and y'' satisfy the congruence $\nu_1'+\mu_1'\equiv\nu_1+\mu_1+y''(1+c_1+d_1)+x''(1+a_1+b_1)+2^{m-6}\{\kappa[a_1(x''-\theta_{x''})+x''y''-\theta_{x''y''}]+\beta_2x''\theta_{y''}+(\kappa+\beta_2)[d_1(y''-\theta_{y''}+2y''(b_1x''+d_1y''))]\} \pmod{2^{m-4}}$.

(G_1) $b_1=2^{m-5}b_2$, $d_1=2^{m-5}d_2$, $a_1=c_1=\mu_1=0$. $(G_1)\sim$ the five types given below and $C=\begin{bmatrix}Q, & P, & R\\ P, & Q, & R\end{bmatrix}$ and $\begin{bmatrix}P, & QP^{2^{m-6}x_1'}, & RQ^{2y_1''}P^{2x_1''}\\ P, & Q, & R\end{bmatrix}$, where x_1', x_1'' and y_1'' satisfy $x_1'+\beta_2'-\beta_2+2^{m-6}x_1'(\kappa+\beta_2)\equiv 0 \pmod 4$, $y_1''+x_1''\equiv 0 \pmod{2^{m-5}}$, $b_2'+b_2+\kappa y_1''\equiv 0 \pmod 2$, and $d_2'+d_2+(\kappa+\beta_2)x_1''\equiv 0 \pmod 2$.

(G_2) $b_1=-1+2^{m-6}b_2$, $d_1=-1+2^{m-6}(2d_3+b_2)$, $a_1=c_1=1$, $\mu_1=0$. $(G_2)\sim(G_1)$ and $C=\begin{bmatrix}P, & Q, & RQ^{y''}P\\ P, & Q, & R\end{bmatrix}$, where $y''=-1+2^{m-6}y_1''$ and satisfies $y_1''+2d_3+2^{m-6}b_2y_1''+\beta_2(-1+2^{m-6}y_1'')\equiv 0 \pmod 4$. There are five types in (G). They are defined as follows:

$Q^{-1}PQ=Q^2P^{-1}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^{-1}PR=P^{1+2^{m-4}b_2}$, $R^{-1}QR=QP^{2^{m-4}d_2}$, $R^2=1$, $Q^4=P^4$, $P^{2^{m-3}}=1$. $\kappa=0$; $d_2=0$, $b_2=0$, 1, $b_2=d_2=1$; $\kappa=1$, $d_2=0$, $b_2=0$, 1.

(G') $a=1+2a_1$, $b=2b_1$. The groups for μ and ν odd $\sim$ those for μ and ν even, with $C=\begin{bmatrix} P, & Q, & RP \\ P, & Q, & R \end{bmatrix}$. Transformation of (1), (2), (3), (4), (5), and $Q^4=P^4$ by R, gives $c=2c_1$, $d=1+2d_1$, $\beta_2=0$, and the congruences

$a_1+b_1\equiv c_1+d_1 \pmod{2^{m-5}}$...(10),

$\mu_1\equiv\nu_1 \pmod 2$...(11),

$\mu_1(c_1+d_1)+\nu_1(a_1+b_1)\equiv 0 \pmod{2^{m-5}}$...(12),

$b_1+c_1\equiv 0 \pmod 2$...(13),

$a_1+b_1+c_1+d_1+2[a_1(b_1+c_1+d_1)+b_1^2]+2^{m-5}\kappa(a_2c_1+\mu_1)\equiv 0 \pmod{2^{m-4}}$.(14),

$a_1+b_1+c_1+d_1+2[d_1(a_1+b_1+c_1)+c_1^2]+2^{m-5}\kappa(\nu_1+b_1d_1)\equiv 0 \pmod{2^{m-4}}$ (15),

$[1, y, x]^2=[0, 2\mu_1+y(1+2c_1)+x(1+2a_1)+2y\theta_{x+y}, 2\nu_1+x(1+2b_1)+(-1)^{x+y}y$
$+2d_1y+2^{m-4}(\kappa+\beta_2)\{(x+y)\nu_1+b_1xy\}+2^{m-5}\{\kappa[b_1(x-\theta_x)+c_1(y-\theta_y)+$
$y(x+y)(1+2d_1)+2a_1xy-\theta_{y(x+y)}]+\beta_2[b_1(x-\theta_x)+y\theta_{x+y}(1+2d_1)]\}]$...(16).

The groups for ν_1 other than $0\sim$ those for $\nu_1=0$, through $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$ where $z''=1$, and y'' and x'' satisfy $2\mu_1+y''(1+2c_1)+x''(1+2a_1)+2y''\theta_{x''+y''}+2\nu_1+x''(1+2b_1)+(-1)^{x''+y''}y''+2d_1y''+2^{m-4}(\kappa+\beta_2)[\nu_1(x''+y'')+b_1x''y'']+2^{m-5}\{\kappa[b_1(x''-\theta_{x''})+c_1(y''-\theta_{y''})+y''(x''+y'')(1+2d_1)+2a_1x''y''-\theta_{y''(x''+y'')}]+\beta_2[b_1(x''-\theta_{x''})+y''\theta_{x''+y''}(1+2d_1)]\}\equiv 2(\mu_1'+\nu_1') \pmod{2^{m-3}}$. The groups for $a_1+b_1\equiv 0 \pmod{2^{m-5}}\sim$ those, where $a_1=b_1=0$, with $C=\begin{bmatrix} P, & Q^{1+2a_1}P^{2b_1}, & R \\ P, & Q, & R \end{bmatrix}$.

(G_1') $b_1=2^{m-6}b_3$, $d_1=-2^{m-6}b_3$, $a_1=c_1=0$.

(G_2') $b_1=-1+2^{m-6}b_3$, $d_1=-2-2^{m-6}b_3$, $a_1=0$, $c_1=1$.

(G_3') $b_1=-2+2^{m-6}b_3$, $d_1=-1-2^{m-6}b_3$, $a_1=1$, $c_1=0$.

(G_4') $b_1=-1+2^{m-6}b_3$, $d_1=-1+2^{m-6}(b_3+2d_3)$, $a_1=c_1=1$.

(G_3') and $(G_4')\sim(G_2')$ and (G_1'), respectively, with $C=\begin{bmatrix} Q^{2y_1}P^x, & Q^{y'}Q^{2x_1'} \\ P, & Q \end{bmatrix}$ where x and y' are odd and the variables satisfy $x_1+y_1-x_1'+y_1'\equiv 0 \pmod{2^{m-5}}$, $a_1'+y_1'\equiv a_1+x_1 \pmod 2$, $x_1+y_1-x_1'-y_1'+a_1-a_1'+b_1-b_1'+2x_1(a_1+b_1-b_1')+2y_1(c_1+d_1-b_1')-2a_1'(x_1'+y_1')+2^{m-5}\kappa\{a_1x_1'+b_1x_1+y_1(1+c_1+b_1')\}\equiv 0 \pmod{2^{m-4}}$, $c_1'+y_1\equiv c_1+x_1' \pmod 2$, $x_1+y_1-x_1'-y_1'+c_1'-c_1+d_1'-d_1+2d_1'(x_1+y_1)+2c_1'(x_1'+y_1')-2y_1'(c_1+d_1)-2x_1'(a_1+b_1)+2^{m-5}\kappa\{c_1'x_1'+d_1'y_1+x_1'(1+b_1)+c_1y_1'\}\equiv 0 \pmod{2^{m-4}}$. The groups in (G_1') and (G_2') are simply isomorphic with the types given below, by $C=\begin{bmatrix} Q, & P, & R \\ P, & Q, & R \end{bmatrix}$, for (G_1) and for

$\kappa=0$ in (G_2); and $C=\begin{bmatrix} P, & QP^{2^{m-4}}, & R \\ P, & Q, & R \end{bmatrix}$, for $\kappa=1$, in (G_2'). There are four types in (G_1') and four in (G_2'). Their defining equations are:

$Q^{-1}PQ=Q^2P^{-1}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^2=1$, $Q^4=P^4$, $P^{2^{m-3}}=1$, $\kappa=0, 1$.

(G_1), $R^{-1}PR=QP^{2^{m-5}b_1}$, $R^{-1}QR=P^{1-2^{m-5}b_1}$; (G_2), $R^{-1}PR=QP^{-2+2^{m-5}b_1}$,

$R^{-1}QR=Q^2P^{-1-2^{m-5}b_1}$, $b_1=0, 1$.

§2. $\{P, Q\}$ IS OF ORDER 2^m.

$\{P, Q\}=G_m$. $\{P\}=G_{m-3}$. Two cases arise, viz: (A) Q^2 is in G_{m-2}; (B) Q^2 is not in G_{m-2}.

(A) *Q^2 is in G_{m-2}.* Here $Q^{-2}PQ^2=P^{\omega+2^{m-4}\kappa}$ ...(1), and $Q^4=P^{4\lambda}$...(2). From (1), $[0, -2y_1, x, 0, 2y_1]=[0, 0, (\omega)^{y_1}(1+2^{m-4}\kappa y_1)x]$...(3), and $[0, 2y_1, x]^s=[0, 2sy_1, \{\frac{1+(\omega)^{y_1}}{2}+2^{m-5}\kappa y_1\}(s-\theta_s)x+x\theta_s]$...(4). Let R be some operator in G_{m-1}, not G_{m-2}. Then $G_{m-1}=\{R, G_{m-2}\}$. Also R^2 is in G_{m-2} and in $\{P\}$, for otherwise, $G_{m-1}=\{R, P\}=\{Q', P\}$ which has been considered in §1. Hence $R^2=P^{2\mu}$...(5). In G_{m-1}, $R^{-1}PR=Q^{2a}P^b$...(6), $R^{-1}Q^2R=Q^{2c}P^d$...(7).

(A_1) *$\omega=1$.* From (6) by (4), $b=1+2b_1$. Now (RP^4) and $(RQ^2)^4$ are in $\{P\}$. Hence $a=0$, $c=1$, $d=2d_1$. Then $\{R, P\}$ is of order 2^{m-2} and may be reduced to five cases. $R^{-1}PR=P^{\omega_1+2^{m-4}b_2}$...(6), $R^2=P^{2^{m-4}\mu_1}$...(5). $\omega_1=\pm 1$, $b_2=0, 1$, $\mu_1=0$; $\omega_1=-1$, $b_2=0$, $\mu_1=1$. From (6), $[-y, 0, x, y]=[0, 0, (\omega_1)^y(1+2^{m-4}b_2)x]$...(8), and $[y, 0, x]^s=[sy, 0, \{\frac{1+(\omega_1)^y}{2}+2^{m-5}b_2y\}(s-\theta_s)x+x\theta_s]$...(9). Transform (2) and (7) by R. There result $\lambda(1-\omega_1)+d_1\equiv 0 \pmod{2^{m-5}}$...(10), $d_1(1+\omega_1)\equiv 0 \pmod{2^{m-4}}$...(11). $[1, 2y_1, x]^2=[0, 0, 2^{m-4}\mu_1+4\lambda y_1+2d_1y_1+x\{\omega_1+1+2^{m-4}(\kappa y_1+b_2)\}]$...(12). In G_m, $Q^{-1}PQ=RQ^{2f}P^g$...(13), $Q^{-1}RQ=RQ^{2h}P^n$ or $Q^{2h}P^n$...(14). For the second alternative of (14), the relations are inconsistent. We now transform the defining relations by Q. For $\omega_1=-1$, these results show that the relations are inconsistent. Let $Q'=QP^x$ (x odd and even). There result $d=2^{m-5}d_2$, $g=1+2g_1$, $n=2^{m-4}n_2$, $h=0$ and $\lambda=0$ except where $g_1=-1+2^{m-6}g_2$,—g_2 odd, $f=1$, g_2 odd or even, $f=0$,— when $\lambda=2^{m-6}\lambda_1$, $(\lambda_1=0, 1)$; and the congruences

$g_1=2^{m-6}g_2$, $n_2+g_2\equiv\kappa \pmod 2$...(15); and

$g_1=-1+2^{m-6}g_2$, $\kappa f+b_2+n_2+g_2\equiv\kappa \pmod 2$...(16).

$[0, y, 0, z, 0, x]^s=[0, sy+(s-\theta_s)f\theta_{xy}, 0, sz+\frac{s-\theta_s}{2}xy, 0, sx+2^{m-5}s(s-1)x\{b_2xy+$

$\kappa(f\theta_{xy}+y)\}+(s-\theta_s)\{g_1x\theta_y+2^{m-5}[(b_2x+n_2y)z+\kappa x(\frac{y-\theta_y}{2})+y(\frac{x-\theta_x}{2})(b_2$

$+\kappa f+\lambda_1 f)]\}]$...(17).

The groups in (A_1) are simply isomorphic with the types given below, with $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where $P'=Q^y R^z P^x$, $Q'=Q^{y'} R^{z'} P^{x'}$, $R'=Q^{2y_1''} R^{z''} P^{2^{m-5}x_1''}$. In these x and y' are odd; for g_1 even, $x'=2^{m-5}x_1'$; for g_1 odd $y=2y_1$. The variables satisfy $\kappa(1+f\theta_{x'})+\kappa'(1+g_1\theta_y)+b_2x'\equiv 0 \pmod 2$, $\kappa y_1''+z''(n_2y+b_2)\equiv b_2' \pmod 2$, $f\equiv y_1''+f'(1+f\theta_{x'}) \pmod 2$, $1\equiv z''+f'x' \pmod 2$, $2^{m-5}\{\lambda_1[f(1+x_1)+f'(1+f\theta_{x'})-y_1''] + \kappa[x'y_1+y_1'+f(x'+x_1)] + b_2[x'(1+z)+z'+x_1]+n_2[y(z'-z'')+z] + f'\theta_y[n_2z'+x_1'] + g_1'[b_2z+\kappa y_1]\} - 2^{m-6}x_1'' + x[g_1-g_1'(1+g_1\theta_y)]\equiv 0 \pmod{2^{m-4}}$, $x'(\kappa y_1''+b_2z'')+g_2x_1''+n_2z''\equiv n_2' \pmod 2$, $x'(1+g_1)+2^{m-6}\{\lambda_1(1+fx')-\lambda_1'\}\equiv 0 \pmod{2^{m-5}}$, $\lambda_1y_1''+x_1''\equiv 0 \pmod 2$.

$(A_{\bar{1}})$ $\omega=-1$. The relations when transformed by Q are found to be inconsistent. There are therefore sixteen types in (A), viz:

$$Q^{-1}PQ=RQ^{2f}P^{\omega_2+2^{m-5}g_2},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=P^{1+2^{m-4}b_2},\quad Q^{-1}RQ=RP^{2^{m-4}n_2},\quad R^2=1,\quad Q^4=P^{2^{m-4}\lambda_1},\quad P^{2^{m-3}}=1.$$

κ	b_2	f	g_2	n_2	λ_1	ω_2	κ	b_2	f	g_2	n_2	λ_1	ω_2	κ	b_2	f	g_2	n_2	λ_1	ω_2
0	0,1	0	0	0	0	1	1	0	0	0	1	0	1	1	0	0	1	0	0	-1
1	1	0	1	0	0		0	0	0	0	0	0,1	-1	0	1	0	0	1	0,1	
1	0	0	1	0	0		1	0	0	0	1	0,1		1	0	1	1	1	0	
0	0	0	1	1	0		0	0	0	1	1	0		0	1	0	1	0	0	

(B) *Q^2 is not in G_{m-2}.*

$R^{-1}PR=P^{\omega_1+2^{m-4}\kappa}$ ...(1),

$R^2=P^{2^{m-4}\mu}$ ...(2).

From (1), $[-y, 0, x, y]=[0, 0, \{(\omega_1)^y+2^{m-4}\kappa y\}x]$...(3), and

$[y, 0, x]^s=[sy, 0, \{\frac{1+(\omega_1)^y}{2}+2^{m-5}\kappa y\}(s-\theta_s)x+x\theta_s]$...(4).

In G_{m-1}, $Q^{-2}PQ^2=R^a P^b$...(5).

From (5) by (4), $b=1+2b_1$. $(Q^2P)^2=Q^4R^aP^{2b}$. If $a=1$, Q^2P would be an operator Q' of G_{m-1}, where $Q'^2\neq\{P\}$, which was discussed in §1. If $a=0$, $Q^{-2}PQ^2=P^{1+2b_1}$, and Q^2 is in $G'_{m-2}=\{Q_1^2P\}$, which was discussed in (A). Hence there are no new types in (B).

PART 3. THE SQUARE OF EVERY OPERATOR IS IN $\{P\}$.

The group G_{m-1} is determined from the types in §3, Part 2, by replacing m by $m-1$. The types of G_{m-1} are

(A) $Q^{-1}PQ=P^{1+2^{m-4}\kappa}$ ...(1), $R^{-1}PR=P^{\omega_1+2^{m-4}\beta_1}$...(2), $R^{-1}QR=QP^{2^{m-4}b_1}$

$\ldots(3)$, $R^2=1$, $Q^2=1$, $\omega_1=\pm1$, $\kappa=0$, $\beta_1=0, 1$, $b_1=0$; $\beta_1=0$, $b_1=1$; $\omega_1=-1$, $\kappa=\beta_1=1$, $b_1=0, 1$.

(*B*) $Q^{-1}PQ=P\ldots(1)$, $R^{-1}PR=P^{-1}\ldots(2)$, $R^{-1}QR=Q\ldots(3)$, $R^2=P^{2^{m-4}}$, $Q^2=1$.

Let S be an operator in G_m, not in G_{m-1}.

Then $S^2=P^{2\nu}\ldots(4)$.

In $G_m=[S, G_{m-1}]$, $S^{-1}PS=R^aQ^bP^c\ldots(5)$,

$S^{-1}QS=R^eQ^fP^g\ldots(6)$,

$S^{-1}RS=R^hQ^iP^j\ldots(7)$.

$(SP)^2$, $(SQ)^2$, and $(SR)^2$ are in $\{P\}$; hence $a=b=e=i=0$, $c=\omega+2^{m-4}c_1$. From (6), $g=2^{m-4}g_1$. Transformation of (7) by S shows $j=2^{m-4}j_1$. In all cases $\nu=0$, except when $c=-1$, when $\nu=2^{m-4}\nu_1$.* The groups in Part 3 are simply isomorphic with the following types:

$Q^{-1}PQ=P$, $R^{-1}PR=P^{\omega_1+2^{m-4}\beta_1}$, $R^{-1}QR=Q$, $S^{-1}PS=P^{\omega+2^{m-4}c_1}$, $S^{-1}QS=QP^{2^{m-4}g_1}$, $S^{-1}RS=RP^{2^{m-4}j_1}$, $S^2=P^{2^{m-4}\nu_1}$, $R^2=1$, $Q^2=1$, $P^{2^{m-3}}=1$.
$\omega=\omega_1=1$, $\beta_1=g_1=\nu_1=0$, $c_1=0$, $j_1=0, 1$; $c_1=1, j_1=0$; $\omega_1=1$, $\omega=-1$, $g_1=0$; $\nu_1=0$; $\beta_1=0$, $c_1=0, j_1=0, 1$; $\beta_1=0$, $c_1=1, j_1=0$; $\beta_1=1$, $c_1=0$, $j_1=0$; $\nu_1=1$, $\beta_1=c_1=j_1=0$; $\omega_1=-1$, $\omega=1$, $\beta_1=0, 1$, $g_1=1$, $c_1=j_1=0$.

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*Paragraph 2.

A METHOD OF APPROXIMATION.

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The following method of approximation may or may not be new, but as I believe it to be of practical importance I wish to call attention to it, and to point out that to secure the same degree of accuracy, it involves, at least in some cases, less labor than does the method of mechanical quadrature. It can also be used in certain cases where the method of mechanical quadrature fails, because the finite differences used in the latter method cannot always be found between the proper limits. On account of its rapid convergence it can also be used when other common methods fail.

We have the formula,*

$$f(x)=f(0)+\frac{x}{2m}\left\{f'(x)+f'(0)+2\left[f'\left(\frac{x}{m}\right)+f'\left(\frac{2x}{m}\right)+\dots+f'\left(\frac{m-1}{m}x\right)\right]\right\}$$

$$-\frac{B_1x^2}{m^2.2!}[f''(x)-f''(0)]+\frac{B_2x^4}{m^4.4!}[f^{iv}(x)-f^{iv}(0)]-\frac{B_3x^6}{m^6.6!}[f^{vi}(x)-f^{vi}(0)]+\dots$$

$$+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}[f^{2n}(x)-f^{2n}(0)]+\dots\quad\dots(1),$$

(B_1, B_2, B_3, etc., being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, $\frac{5}{66}$, etc.)

For brevity, (1) may be written,

$$f(x)=F(x,m)+\frac{s_2}{m^2}+\frac{s_4}{m^4}+\frac{s_6}{m^6}+\text{ etc. }\dots(2),$$

in which s_2, s_4, s_6, etc., are independent of m. It is evident that m may be considered a convergence factor which makes $F(x, m)$ approach $f(x)$ as m approaches infinity, F and f becoming equal as m becomes infinite. But as m increases, the labor of evaluating $F(x, m)$ also increases, so that, in practice, it is found desirable to obtain one or more of the higher derivatives involved in s_2, s_4, s_6, etc., in order to permit the use of a smaller value of m, and yet to attain the required degree of accuracy. In certain cases it becomes exceedingly difficult to obtain the form and value of the higher derivatives involved in s_2, s_4, s_6, etc., and chiefly for this reason, it becomes desirable to eliminate from (2) certain of these quantities, (s_2, s_4, s_6, etc.), after a certain degree of approximation has been reached. To accomplish this elimination take more than one value of m and set down a separate equation for each value chosen, thus,

*See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

$$f(x)=F(x,\, m_1)+\frac{s_2}{m_1^2}+\frac{s_4}{m_1^4}+\frac{s_6}{m_1^6}+\ldots \quad \ldots(3),$$

$$f(x)=F(x,\, m_2)+\frac{s_2}{m_2^2}+\frac{s_4}{m_2^4}+\frac{s_6}{m_2^6}+\ldots \quad \ldots(4),$$

$$f(x)=F(x,\, m_3)+\frac{s_2}{m_3^2}+\frac{s_4}{m_3^4}+\frac{s_6}{m_3^6}+\ldots \quad \ldots(5),$$

etc. (In selecting the values of m it is usually convenient to take the highest value such that one or more of the lesser values will be sub-multiples thereof.)

It is evident that from $(r+1)$ such linear equations there may be eliminated r of the quantities s, say s_{2i}, $s_{2(i+1)}$, ..., $s_{2(i+r-1)}$. There results,

$$f(x)=\frac{\begin{vmatrix} M_1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots & m_1^{-2(i+r-1)} \\ M_2 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots & m_2^{-2(i+r-1)} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ M_{(r+1)} & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}{\begin{vmatrix} 1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots & m_1^{-2(i+r-1)} \\ 1 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots & m_2^{-2(i+r-1)} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 1 & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}} \quad \ldots(6),^*$$

where M_1, M_2, ..., $M_{(r+1)}$, represent the approximate value of $f(x)$ obtained by (2) for m_1, m_2, ..., $m_{(r+1)}$, respectively, by the use of all the terms preceding s_{2i}/m^{2i}.

If all derivatives higher than the $(2i+2r-1)$th are zero, or if all such *even-numbered* derivatives are equal for $x=x$ and $x=0$, (6) gives the exact value of $f(x)$. In other cases the degree of accuracy attained will be no less than would have been obtained by taking the smallest value of m used in (6) and developing $f(x)$ by (2) so far as to include all the r terms eliminated in (6).

As a special case of (6), let $r=2$, $i=2$. Here $M=F(x,\, m)+\frac{s_2}{m^2}$, and we get after reducing,

$$f(x)=\frac{m_1^6(m_2^2-m_3^2)M_1+m_2^6(m_3^2-m_1^2)M_2+m_3^6(m_1^2-m_2^2)M_3}{m_1^6(m_2^2-m_3^2)+m_2^6(m_3^2-m_1^2)+m_3^6(m_1^2-m_2^2)} \ldots(7).$$

*Determinants of this form may be readily evaluated. See Weld's *Theory of Determinants*, Articles 23 and 27.

It is clear that the more nearly all the M's approach $f(x)$, the more nearly does (6) give the exact value of $f(x)$, and that this increase in approximation is obtained by sufficiently increasing the integers, m, i, and r, or any of them.

The following points may be noted in passing:

First. After choosing certain values of m, r, and i, $f(x)$ is developed by (6) in linear terms of the $(r+1)$ quantities $(M_1, M_2, \ldots, M_{(r+1)})$, each M being an approximate value of $f(x)$.

Second. If $M=F(x, m)$ and if the values (but not the form) of $f'(x)$ be given at each of the $(m+1)$ equidistant points $(0, x/m, 2x/m, \ldots, x)$ for each of the $(r+1)$ values of m chosen, the value of $f(x)$ may be very closely approximated, although in such case the law of its development would not be expressly stated, but it must be assumed that some such law exists in order that $f(x)$ may be developable in terms of x. This makes the method of practical value in many scientific problems where it becomes necessary to find $f(x)$ approximately, and when the only available data are a number of observations of the values of $f'(x)$ at intervals of x/m, the approximation being closer in this method than in that of mechanical quadrature, especially where the intervals (x/m) are few.

A discussion of the cases where (6) does not give the value of $f(x)$ is not here entered into, but the following considerations will be pertinent:

First. As (6) is obtained from a number of equations, (3), (4), (5), etc., care should be taken that each of these equations actually does give a closer approximation of $f(x)$, when the number of terms employed include those eliminated in deriving (6), than does M.

Second. As (6), (2), and (1) are all dependent on the sum of a number of series developed by Stirling's formula, it becomes necessary that each of these underlying Stirling's series should give a true and convergent development of f between any and all of the adjacent, equidistant points (x/m) at which the value of f' is taken.

The following example has been chosen because its development by formula (1) has been given in the MONTHLY,* Vol. XIII, No. 4, and because its solution by mechanical quadrature has been given by Dr. G. W. Hill in the *Analyst*, Vol. II, p. 120, 1875.†

Example. Evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{xdx}{\sin x(1+.16\cos^2 x)^{\frac{3}{2}}}$.

Taking $r=2$, $i=2$, $m_1=6$, $m_2=3$, $m_3=2$, we get $M_1=1.657,626,355$, $M_2=1.657,490,406$, $M_3=1.657,013,853$. Substituting in (7), we get,

*The value of the integral is incorrectly given in the MONTHLY as, 1.657,636,524, instead of 1.657,636,257. The error is due to the fact that

$$f^{vi},\ \left[\frac{2596\pi^6}{4200.12^6.6\,!}\right]$$

is carried out as .000,000,009 instead of .000,000,276.

†See also Hill's *Collected Works*, Vol. I, p. 204.

$$f(\tfrac{1}{2}\pi)-f(0)=\int_0^{\frac{1}{2}\pi}\frac{xdx}{\sin x(1+.16\cos^2x)^{\frac{3}{2}}}=\frac{1728M_3-23328M_2+233280M_1}{211,680}$$
$$=1.657,636,33...(8).$$

We may check the accuracy of the work of computation and determine the degree of approximation attained in (8) by finding the values of $f'(\frac{\pi}{8})$ and $f'(\frac{3\pi}{8})$. This done we may take $m_1=6$, $m_3=4$, $m_2=3$. Substituting these values of m_1, m_2, m_3, in (7) we get an expression similar to (8) which gives a value of the definite integral coinciding with the value found in (8) for six decimal places, thus proving the accuracy of the work and showing the number of decimal places to which result found by (8) is correct.

This result is not as accurate as that obtained in the April MONTHLY for the reason that the *smallest* value of m here used is 2. This is therefore a more accurate result than would have been obtained by taking $m=2$ and developing by (1) so far as to include the term involving B_4, but a less accurate result than was obtained in the MONTHLY by taking $m=6$ and developing by (1) far enough to include the term involving B_3. The above result is, however, as accurate as Dr. Hill's and was obtained with much less labor. To obtain a more accurate result than that given in the MONTHLY without finding any higher derivatives than are there given, take $i=4$, $r=1$, $m_1=6$, $m_2=3$, and substitute in (6). By using nine decimals throughout the result is found to be 1.657,636,259. By using ten decimals throughout a result correct to nine or ten decimal places would have been obtained.

APPROXIMATION OF THE GREATEST ROOT OF A CUBIC EQUATION WITH THREE REAL ROOTS.

By CHARLES GILPIN, JR., Philadelphia, Pa.

We are concerned with the "irreducible case," in which Cardan's formula is of no value for computation. By replacing x by $-x$ if necessary, we need consider only the form

$$x^3-ax-b=0 \quad (a \text{ and } b \text{ positive}) \quad ...(1),$$

the two lesser roots of which are negative and the greatest root, which equals the sum of the lesser ones, is positive. It can be shown* that the greatest root g lies between the limits $\sqrt{a}$ and $\sqrt{(4a/3)}$. From equation (1) we obtain

*$g>\sqrt{a}$ by (2). Since $b^2/4-a^3/27<0$ in the irreducible case, x^3-ax-b is positive for $x=\sqrt{(4a/3)}$, negative for $x=\sqrt{a}$. EDITOR.

$$x=\sqrt{a+\frac{b}{x}}\ldots(2).$$

It is evident that if we assume for g an approximate value less than the true one, and substitute it in the right hand member of equation (2), the resulting value will be greater than the true value; if we assume for g a value greater than the true one, the resulting value will be less than the true value.

For a first approximation to the value of g, assume any convenient positive value between the limits $\sqrt{a}$ and $\sqrt{(4a/3)}$. For a second approximation, substitute the first approximation in the right hand member of equation (2) and compute the resulting value; etc.

This will give a series of approximations, alternately less and greater than the true value, towards which they converge as a limit.

Example. $x^3-6x-2=0$.

$$\sqrt{6}=2.449,$$

$$\sqrt{6+\frac{2}{2.449}}=2.610,$$

$$\sqrt{6+\frac{2}{2.610}}=2.601,$$

$$\sqrt{6+\frac{2}{2.601}}=2.6019,$$

$$\sqrt{6+\frac{2}{2.6017}}=2.601677,$$

the last being correct to five decimals.

ON THE FORMULA FOR THE AREA OF A CURVE IN POLAR CO-ORDINATES.

By JACOB WESTLUND, Purdue University.

In deriving the formula $A=\frac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2 d\theta$ for the area between a curve and two radii vectores it is customary to consider the area as the limit of the sum of infinitesimal circular sectors. This formula* may, however, be derived directly from the formula $A=\int_{x_1}^{x_2} ydx$ for the area between a curve, the axis of x, and two

*It is very probable that this desirable method of proof occurs in the literature; it has been in use in Professor E. H. Moore's course in Calculus. Ed. D.

ordinates. To do this we assume that the curve may be divided into a finite number of parts such that for any part the abscissa x either increases or decreases constantly when we go from one extremity of the arc to the other. Let us then consider an arc AB, where x and ρ increase when θ increases from θ_1 to θ_2. Let OP be the initial line, A', B' the feet of the perpendiculars on OP from A and B, respectively, and let $OA=\rho_1$, $OB=\rho_2$.

Now we have

$$OAB=OBB'-OAA'-AA'B'B=\tfrac{1}{2}\rho_2^2\sin\theta_2\cos\theta_2-\tfrac{1}{2}\rho_1^2\sin\theta_1\cos\theta_1-\int_{x_1}^{x_2}ydx.$$

Hence

$$OAB=\tfrac{1}{2}\rho_2^2\sin\theta_2\cos\theta_2-\tfrac{1}{2}\rho_1^2\sin\theta_1\cos\theta_1-\int_{\theta_1}^{\theta_2}\rho\sin\theta(\cos\theta\frac{d\rho}{d\theta}-\rho\sin\theta)d\theta$$

since $x=\rho\cos\theta$ increases continuously in the interval (θ_1, θ_2) and has a continuous derivative.

But by integration by parts we have

$$\int_{\theta_1}^{\theta_2}\rho^2\,\frac{d(\sin\theta\,\cos\theta)}{d\theta}d\theta=\rho_2^2\sin\theta_2\cos\theta_2-\rho_1^2\sin\theta_1\cos\theta_1-2\int_{\theta_1}^{\theta_2}\rho\sin\theta\cos\theta\frac{d\rho}{d\theta}d\theta.$$

Hence

$$\begin{aligned}OAB&=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2\,\frac{d(\sin\theta\,\cos\theta)}{d\theta}d\theta+\int_{\theta_1}^{\theta_2}\rho^2\sin^2\theta\,d\theta\\&=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2(\cos^2\theta-\sin^2\theta)d\theta+\int_{\theta_1}^{\theta_2}\rho^2\sin^2\theta\,d\theta\\&=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2d\theta.\end{aligned}$$

In exactly the same way we prove the formula for the case when x decreases through the interval (θ_1, θ_2). If we now consider any curve which may be divided into a finite number of parts, each part having the property given above, it is easily seen that the area included between the curve and two radii vectores is given by the formula, Area$=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2d\theta$.

*KINEMATIC GEOMETRY. INVERSION AND INVERSORS.

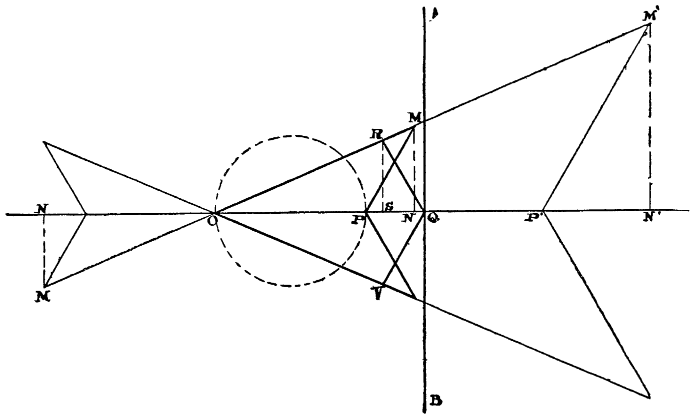
By JOHN JAMES QUINN, Ph. D.

This paper is in the nature of a supplement to the one read by the author before Section A, American Association for the Advancement of Science, in Philadelphia, 1904. In that communication two types of Inversors were exhibited and demonstrated. They represented two distinct groups each embodying the property of inversion, and for special cases they become instruments for describing a line which is mathematically straight.

Further investigation has revealed the fact that an infinite variety of those instruments can be made possessing this principle, differing somewhat in appearance from one another yet essentially the same.

The manner in which they can be constructed is set forth in the following theorems:

THEOREM. *If from any point P in the axis of symmetry OQ of a concave or convex kite lines be drawn parallel to the shorter sides, and terminated by the longer sides (produced if necessary) the product of the distances $OP \times OQ$ is constant, whether the point P be taken within or without the points O and Q.*



GIVEN: The kite $ORQV$; OR produced to M; $MP \parallel QV$; MN and RS perpendicular to OQ.

PROOF. $OP=ON \pm PN=\sqrt{(OM^2-MN^2)} \pm \sqrt{(MP^2-MN^2)}$,

$OQ=OS \pm QS=\sqrt{(OR^2-RS^2)} \pm \sqrt{(RQ^2-RS^2)}$.

Now $RS=\frac{RO}{MO}.MN$; and $RQ=\frac{RO}{MO}.MP$. Substituting we get

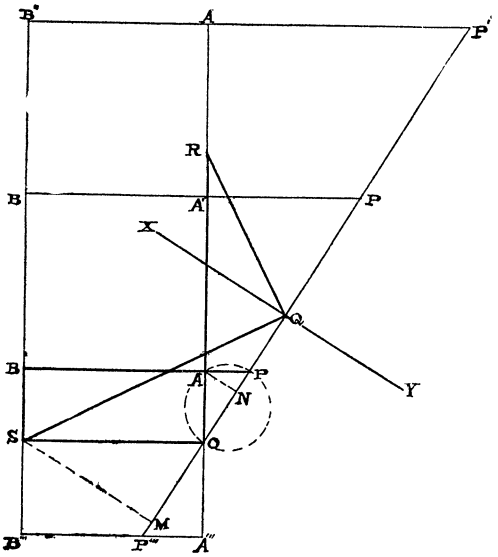
$$OP.OQ=\frac{RO}{MO}[OM^2 \pm MP^2], \text{ a constant.}$$

*Read before Section A, American Association for the Advancement of Science, at the New Orleans meeting, December, 1905.

Therefore the points O, P, and Q are inverse points. Similarly if the points P and Q be interchanged. Q. E. D.

SCHOLIUM. From the above it is evident that if the point O be fixed in position, and the points P, P', etc., be constrained to move in a circle through O, the point Q will describe a straight line, as AB.

THEOREM. *If O and Q be two adjacent vertices of a crossed parallelogram $OQRS$, and P be a point situated on a parallel to OS collinear with O and Q, then $OP \times OQ$ is constant, whether P be within or without the points O and Q.*



GIVEN: The crossed parallelogram $QROS$; $BA \parallel SO$; P on BA produced collinear with O and Q; AN and SM perpendicular to OP.

PROOF. $OP = ON \pm NP = \sqrt{(AO^2 - AN^2)} \pm \sqrt{(AP^2 - AN^2)}$.

$OQ = MQ \pm MO = \sqrt{(QS^2 - SM^2)} \mp \sqrt{(SO^2 - SM^2)}$.

Now $MS = \frac{OS}{AP} . AN$ and $QS = \frac{OS}{AP} . AO$.

Therefore $OP . OQ = \frac{OS}{AP} [AO^2 \mp AP^2]$ is a constant.

Hence the points O, P, and Q are inverse points. Q. E. D.

Similarly, is the points P', P'', etc., be taken.

Evidently then if the point O be fixed and P be constrained to move in a circle through O, the point Q will move in a straight line, as XY. The position of the line described by Q depends upon the position of the center of the circle described by the point P. It is perpendicular to the line connecting O to the center of the circle.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\frac{n}{(4n^2-1)^2}$, beginning with $n=1$, n being always odd.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{(4n^2-1)^2}=\tfrac{1}{8}\left(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right).$$

When $n=1, 3, 5, 7, \ldots$

$$\Sigma\frac{n}{(4n^2-1)^2}=\tfrac{1}{8}\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\ldots-\frac{1}{3^2}-\frac{1}{7^2}-\frac{1}{11^2}-\ldots\right)$$

$$=\tfrac{1}{8}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)-\tfrac{1}{4}\left(\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\ldots\right)$$

$$=\frac{\pi^2}{64}-\tfrac{1}{4}(.1579+)=\frac{\pi^2}{64}-\tfrac{1}{4}\cdot\frac{2\pi^2}{125}\text{ nearly, }=\frac{\pi^2}{64}-\frac{\pi^2}{250}=\frac{93\pi^2}{8000}.$$

We may also write

$$\Sigma\frac{n}{(4n^2-1)^2}=\sum_{m=1}^{\infty}\frac{2m-1}{(16m^2-16m+3)^2}=\tfrac{1}{8}\int_0^1\frac{\tan^{-1}x}{x}dx.$$

This series is discussed by William E. Heal in Vol. IX, pp. 47—49, of the MONTHLY.*

Similar approximations were obtained by S. A. Corey, G. W. Greenwood, and J. Scheffer.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals e and π in the form of infinite continued fractions.

*See also an article entitled "Note on the Numerical Transcendants S_n and $s_n = S_n - 1$," by Professor W. Woolsey Johnson, in the current *Bulletin of the American Mathematical Society*.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

According to a method due to Euler the series

$$S=\frac{1}{A}-\frac{1}{B}+\frac{1}{C}-\frac{1}{D}+\frac{1}{E}-\dots$$

may be converted into a continued fraction thus: Putting

$$S_1=\frac{1}{B}-\frac{1}{C}+\frac{1}{D}-\frac{1}{E}+\dots \quad S_2=\frac{1}{C}-\frac{1}{D}+\frac{1}{E}-\frac{1}{F}+\dots$$

$$S_3=\frac{1}{D}-\frac{1}{E}+\frac{1}{F}-\frac{1}{G}+\dots \text{ etc., we get}$$

$$S=\frac{1}{A}-S_1=\frac{1-AS_1}{A};\ \therefore\frac{1}{S}=\frac{A}{1-AS_1}=A+\frac{A^2S_1}{1-AS_1}=A+\frac{A^2}{-A+(1/S_1)},\text{etc.}$$

Thus, $S=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B+\cfrac{C^2}{C-D+\cfrac{D^2}{E-D+\dots}}}}}$ ...(I).

Since $\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$ we get, substituting in (I), $A=1$, $B=3$, $C=5$, $D=1$, etc.,

$$\tfrac{1}{4}\pi=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\dots}}}}$$

To convert the series $\frac{1}{a}-\frac{1}{ab}+\frac{1}{abc}-\frac{1}{abcd}+\dots$ into a continued fraction, we put in (I), $A=a$, $B=ab$, $C=abc$, $D=abcd$, etc., and thus we obtain

$$\frac{1}{a}-\frac{1}{ab}+\frac{1}{abc}-\frac{1}{abcd}+\dots=\cfrac{1}{a+\cfrac{a}{b-1+\cfrac{b}{c-1+\cfrac{c}{d-1+\dots}}}} \quad \dots\text{(II)}.$$

To convert $\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\dots$ into a continued fraction, we have in (II) only to put $-b$, $-c$, $-d$, ... for b, c, d, ... and thus we get

$$\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\dots=\cfrac{1}{a-\cfrac{a}{b+1-\cfrac{b}{c+1-\cfrac{c}{d+1-\dots}}}} \quad \dots\text{(III)}.$$

Putting $a=2$, $b=3$, $c=4$, $d=5$, ... we get

$$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\ldots=\cfrac{1}{2-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{6-\cfrac{5}{7-\ldots}}}}}$$

Hence $e=2+\cfrac{1}{2-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{6-\cfrac{5}{7-\ldots}}}}}$

Also solved by G. W. Greenwood, and G. B. M. Zerr.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant $2(a_0a_4-4a_1a_3+3a_2^2)$ of the binary quartic $a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2+4a_3x_1x_2^3+a_4x_2^4$ in terms of roots of the latter.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It can be shown that, if

$$a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2-4a_3x_1x_2^3+a_4x_2^4$$
$$\equiv a_0(x_1^2+2px_1x_2+qx_2^2)(x_1^2+2p'x_1x_2+q'x_2^2),$$

then $4\theta^3-I\theta+J=0$, where $I=a_0a_4-4a_1a_3+3a_2^2$, and $\theta=a_2-a_0pp'$.

Let β, γ be the roots of $x_1^2+2px_1x_2+qx_2^2=0$, and α, δ the roots of $x_1^2+2p'x_1x_2+q'x_2^2=0$. Then

$$\theta=\frac{a_0}{6}\Sigma\beta\gamma-\frac{a_0}{4}(\beta+\gamma)(\alpha+\delta)=\frac{a_0}{12}(v-w),$$

where $u=(\beta-\gamma)(\alpha-\delta)$, $v=(\gamma-\alpha)(\beta-\delta)$, $w=(\alpha-\beta)(\gamma-\delta)$. The roots of the reduced cubic are therefore,

$$\frac{a_0}{12}(u-v),\ \frac{a_0}{12}(v-w),\ \frac{a_0}{12}(w-u).$$

It is easily found that $u+v+w=0$. Consequently, $\Sigma vw=-\frac{1}{2}\Sigma u^2$, and

$$\frac{I}{4}=-\Sigma\theta_1\theta_2=-\frac{a_0^2}{144}\Sigma(uv-uw+vw-v^2)=\frac{a_0^2}{144}\cdot\tfrac{3}{2}\Sigma u^2.$$

Hence $I=\frac{a_0^2}{24}\Sigma u^2$, where u, v, w have the values given above.

Also solved by G. B. M. Zerr.

CALCULUS.

220. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

To determine the least polygon of n sides that can be described about a given circle.

Solution by the PROPOSER.

Let ϕ_1, ϕ_2, ..., ϕ_n be the successive angles contained between the lines l_1, l_2, ..., drawn from the center to the vertices of the polygon, and the radii (r) drawn to the points of contact of the sides. The area of the right triangle whose angle at the center is ϕ_1, will be

$$\triangle=\tfrac{1}{2}rl_1\sin\phi_1=\tfrac{1}{2}r.r\sec\phi_1.\sin\phi_1=\frac{r^2}{2}\tan\phi_1.$$

Hence the entire area of the polygon is

$$\Sigma\triangle=\frac{r^2}{2}[\tan\phi_1+\tan\phi_2+\ldots+\tan\phi_n].$$

But $\tan\phi_n=\tan[2\pi-(\phi_1+\phi_2+\ldots\phi_{n-1})]=\tan[2\pi-\theta]$, where $\theta=\phi_1+\phi_2+\ldots+\phi_{n-1}$. Thus $u=\tan\phi_1+\tan\phi_2+\ldots+\tan(2\pi-\theta)$ is to be rendered a minimum. Differentiating partially with respect to ϕ_1, we obtain

$$\frac{\partial u}{\partial\phi_1}=\sec^2\phi_1-\sec^2(2\pi-\theta)=0.$$

Hence $\phi_1=2\pi-\theta=\phi_n$.

In like manner we may show that any angle equals the one preceding it. Hence the minimum polygon is regular.

Also solved by G. W. Greenwood, and J. Scheffer.

DIOPHANTINE ANALYSIS.

136. Proposed by A. H. HOLMES, Brunswick, Maine.

Given $7x^2-111=y^2$. Required a value for y greater than unity which shall be a prime integer.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let $y=q$, $x=p$ be two values that satisfy the equation $y^2-Nx^2=-a$, and $y=n$, $x=m$ two values that satisfy the equation $y^2-Nx^2=1$. Then we evidently have $y^2-Nx^2=(q^2-Np^2)(n^2-Nm^2)=n^2q^2+N^2m^2p^2-N(m^2q^2+n^2p^2)=(nq\pm Npm)^2-N(mq+np)^2$. Therefore, we can put $y=nq\pm Npm$, $x=mq\pm np$. Substituting numerical values we have, since $y=8$, $x=3$ satisfy the equation $y^2-7x=1$,

$$n=\tfrac{1}{2}[(8+3\sqrt{7})^r+(8-3\sqrt{7})^r] \text{ and } m=\frac{1}{2\sqrt{7}}[(8+3\sqrt{7})^r-(8-3\sqrt{7})^r],$$

where for r successive values 1, 2, 3, ... may be put. Since $y=1$, $x=4$ satisfy equation $y^2-7x^2=-111$, we have $q=1$, $p=4$, and thus we find $y=n\pm28m$, $x=m\pm4n$. Substituting for r the numbers 2, 3, 4, we get the sets

$y=76$, $y=92$, $y=1217$, $y=1471$, $y=309119$, $y=373633$,
$x=29$, $x=35$, $x=460$, $x=556$, $x=116836$, $x=141220$, etc.

Thus $y=1471$ is the least prime which satisfies the equation.

Also solved by A. H. Bell.

GEOMETRY.

290. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Show that the point (1, 1) is a conjugate point on the locus $x^3+y^3-3xy+1=0$.

I. Solution by the PROPOSER.

If a line through the point (1, 1) making an angle θ with Ox have a point P in common with the locus, the coördinates of P, *i. e.*, $1+r\cos\theta$, $1+r\sin\theta$, where r is the distance of P from the point (1, 1), satisfy its equation. Therefore

$$(1+r\cos\theta)^3+(1+r\sin\theta)^3-3(1+r\cos\theta)(1+r\sin\theta)+1=0,$$
$$3r^2(\cos^2\theta+\sin^2\theta+\cos\theta\ \sin\theta)+r^3(\cos^3\theta+\sin^3\theta)=0.$$

Two values of r are zero, and the point (1, 1) is therefore a double point. But since no real value of θ will make another value of r zero, the point is a conjugate point.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let us take the more general equation $x^3+y^3-3cxy+c^3=0$. Denoting this polynomial by F, we have

$$\frac{\partial F}{\partial x}=3x^2-3cy,\ \frac{\partial F}{\partial y}=3y^2-3cx.$$

Putting each $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ equal to zero, we get $y=x=c$. It is easy to show that $H=\left(\frac{\partial^2 F}{\partial x\partial y}\right)^2-\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2}=-27c^2$, being negative. Hence (c, c) is a conjugate point.

GROUP THEORY.

8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the eight players being paired in each round, each pair to be matched once and but once in the tournament. List the possible programs, different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

Note by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

All possible solutions may be transformed into one of the six types given below. For type F thanks are given to Dr. Dickson. He kindly sent two solutions for comparison, by which was detected the omission of this type from the final tabulation. The results given have since been carefully checked.

A	15	26	37	48	B	15	26	37	48	C	15	26	37	48
	16	25	34	78		16	23	45	78		16	23	45	78
	17	24	35	68		17	28	35	46		17	28	35	46
	18	23	45	67		18	25	36	47		18	25	36	47
	12	38	47	56		12	34	58	67		12	34	57	68
	13	28	46	57		13	24	57	68		13	24	58	67
	14	27	36	58		14	27	38	56		14	27	38	56

D	15	26	37	48	E	15	26	37	48	F	12	38	47	56
	16	23	45	78		16	23	45	78		13	24	58	67
	17	28	35	46		17	24	35	68		14	35	26	78
	18	25	34	67		18	25	34	67		15	46	37	28
	12	36	47	58		12	36	47	58		16	57	48	23
	13	24	57	68		13	28	46	57		17	68	25	34
	14	27	38	56		14	27	38	56		18	27	36	45

Remarks by the PROPOSER.

Dr. Safford has reduced the possible types to A—F, but has not proved that no two of these are equivalent. I shall prove that this is true, and at the same time answer the remaining questions proposed in the problem.

A very obvious remark shows that A is equivalent to no one of the other types. Set $a_1=(15)(26)(37)(48)$, $a_2=(16)(25)(34)(78)$, ... Then $a_1, \ldots, a_7$ together with identity form a commutative group of order 8 generated by a_1, a_2, a_3, with $a_4=a_2a_3$, $a_5=a_1a_2$, $a_6=a_1a_3$, $a_7=a_1a_2a_3$. But the product of the substitutions corresponding to the first and second lines in B (or C, D, E) is $(1472)(3856)$, so that we reach no group; similarly for F.

We proceed to main problem, that of finding the group of all substitutions*

*For types A, C, and D, Dr. Safford found certain of the substitutions in the course of his reductions, and kindly placed them at my disposal.

leaving a given type unaltered. The order of the group is not the same for any two types, so that no two are equivalent.

For type A, the group is triply transitive, of order 1344, generated by $S=(257)(346)$, $T=(36)(58)$, $V=(1537)(46)$. Obvious products of these replace 1 by 1, ..., or 8. Now S and V transform T into $R=(34)(78)$ and $P=(47)(38)$. Obvious products of these five leave 1 fixed and replace 2 by 2, ..., 8; others leave 2 and 3 fixed and replace 3 by 3, ..., 8. Now $R^{-1}TR=(46)(57)$, which is transformed into $(45)(67)$ by P. Hence there exist substitutions leaving 1, 2, 3 fixed and replacing 4 by 4, 5, 6, or 7, but not by 8, since 8 is necessarily fixed by the fifth line of A. When 1, 2, 3, 4 are fixed, all are fixed. Hence the order is 8.7.6.4.

For type C, the group is simply transitive, of order 24, generated by $R_1=(126)(385)$, $S_1=(267)(345)$, $T_1=(13)(24)(57)(68)$. There exist substitutions, leaving C unaltered, of the form $(1)(2x...)$ for $x=2$, 6, 7, but not for $x=3$, 4, 5, 8. When this is verified for $x=3$ and 8, it follows for $x=4$, 5. Thus, if $W=(1)(24...)$ occurred, then would $WS_1^{-1}=(1)(23...)$ occur. Since identity alone leaves 1 and 2 fixed, the order is 8.3.

For type D, the group is simply transitive, of order 96, generated by $(1254)(3678)$, $(13)(57)$, $(24)(68)$, $(537)(648)$, the first transforming the second into $(26)(48)$. Hence there occur substitutions $(1)(2x...)$ for $x=2$, 4, 6, 8. I find that no substitution $(1)(23...)$ leaves D unaltered. Hence this is true for $(1)(25...)$ and $(1)(27...)$, in view of the last generator. If, for y even, $(1)(2)(3y...)$ occurred, then by transforming by $(24)(68)$, $(26)(48)$ or by their product, we would reach $(1)(32...)$, whereas its inverse does not occur. When 1, 2, and 3 are fixed, all are; hence the order is 8.4.3.

For the type E, the group is transitive, of order 64, and is generated by $(1472)(3856)$, $(1876)(2345)$, $(17)(68)$, $(15)(37)$, $(17)(35)$. Combinations of the first two replace 4 by 1, ..., 8. A substitution which leaves 4 fixed must replace 6 by 6 or 8. Now $(4)(6)(5x...)$ occurs if and only if $x=1$, 3, 5, 7. If 1, 4, 6 are fixed, all are. The order is 8.2.4.

For type F, the group is intransitive, of order 42, and generated by (2345678), $(346)(587)$, $(38)(47)(56)$, being the metacyclic group commutative with the cyclic G_7.

For type B it was found that $(28)(46)$ is the only non-identical substitution not altering B and leaving 1 fixed. Another substitution occurring is $(15)(37)$. Hence the group is of order $4n$, $n=1$, 2, 3, or 4. In any event, the order is less than the orders in the previous cases.

The number of conjugate programmes of type A is $8!\div1344=15$.

MISCELLANEOUS.

158. Proposed by **THEODORE L. DeLAND**, Treasury Department, Washington, D. C.

An ingot of pure gold was melted at the Mint and then 10 ounces were taken out and 10 ounces of pure silver added and the contents of the melting pot

mixed thoroughly. This was repeated until there were 10 such operations in all. The contents of the pot being then assayed was found to be nine-tenths fine, or standard gold. What was the weight of the original ingot? There was no loss in the precious metals by the melting.

Solution by the PROPOSER.

Let X=the weight of the original pure gold ingot; $n=10$=the number of operations; U_n=the weight of pure gold in the pot after the nth operation; $c=10$=the weight of metal taken from the pot, and also the weight of the silver put in, at each operation; cU_nX^{-1}=the weight of pure gold taken out at the $(n+1)$th operation; U_{n+1}=the weight of the pure gold in the pot after the $(n+1)$th operation; and $U_n=0.9X$.

Equate the elements defined above and we have:

$$U_{n+1}=U_n-cU_nX^{-1}\ldots(1);\text{ or } U_{n+1}=(1-cX^{-1})U_n\ldots(2).$$

This is an equation in Finite Differences. Integrate it, and we have:

$$U_n=C(1-cX^{-1})^n\ldots(3).$$

Equation (3) is true for all values of n. When $n=0$, $U_n=U_0=X$, and $C=X$. Substitute this value of C in (3) and we have:

$$U_n=X(1-cX^{-1})^n\ldots(4).$$

We have $U_n=0.9X$. Eliminate U_n, supply numerical values, reduce, and we have:

$$(1-10X^{-1})^{10}=0.9\ldots(5).$$

Therefore $1-10X^{-1}=\sqrt[10]{(0.9)}$; or $X=10\div[1-\sqrt[10]{(0.9)}]$.

The log of $0.9=\bar{1}.954242509=\overline{10}+9.9542425094$, this divided by 10 gives $\bar{1}.9954242509$, which is the log of 0.9895192581, this subtracted from 1, gives 0.01048007419, the log of which is $\bar{2}.0203920257$, which subtracted from 1 (the log of 10) gives 2.9796079743, which is the log of $954.1309293=X$, the original weight of the pure gold ingot, in ounces.

Also solved by S. A. Corey, G. W. Greenwood, and J. Scheffer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

269. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the hyperbolic functions of x in the form of infinite continued fractions.

270. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

Two ferry-boats started simultaneously from opposite sides of a river and one being faster than the other, they met 720 yards from the shore. Each boat remained 10 minutes in its slip to change passengers and started on its return trip, when it was found that they met again 400 yards from the other shore. What is the width of the river?

CALCULUS.

221. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Find $\lim\limits_{x \doteq 0} \tan^{-1}x(\log x)$.

222. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

If $s_n=2\left(\frac{1}{n}-\frac{2}{2n^3}+\frac{1}{5n^5}+\frac{1}{7n^7}-\frac{2}{9n^9}+\frac{1}{11n^{11}}+\ldots\right)$ prove that

$$\log 3=s_3+s_4,$$
$$\log 7=s_2+s_3+s_4,$$
$$\log 13=s_2+2s_3+s_4.$$

223. Proposed by O. E. GLENN, Ph. D., Springfield. Mo.

Prove that $\lim\limits_{n \doteq \infty} \dfrac{\sum\limits_{\lambda=1}^{n} \lambda^k}{n^{k+1}}=\dfrac{1}{k+1}$.

224. Proposed by W. J. GREENSTREET, M. A.. Editor of The Mathematical Gazette, Stroud, England.

Prove that $\int_0^\infty \tan^{-1}(\tan\alpha \sin x)\frac{dx}{x}=\frac{1}{2}\pi \log(\tan\alpha+\sec\alpha)$.

GEOMETRY.

295. Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Md.

One side and the opposite angle of a triangle are fixed. Find the locus of the center of the inscribed circle. Solve by methods of analytic geometry.

296. Proposed by J. J. QUINN, Ph. D., Warren, Pa.

Given $AB=BC$ perpendicular to each other, and E and M their mid-points, respectively. On AB describe a semi-circle, and draw CE to meet the circumference in D. Draw DM cutting AB in F. In what ratio is AB divided by the point F?

NOTES AND NEWS.

Miss M. M. Young has been appointed instructor in mathematics at Wellesly College.

At Princeton University Dr. R. L. Moore has been appointed instructor in mathematics.

Mr. Ralph M. Barton has been appointed instructor in mathematics at Dartmouth College.

Dr. J. G. Hardy has been promoted to an associate professorship of mathematics at Williams College.

Mr. J. K. Whittemore has been appointed assistant professor of mathematics at Harvard University.

Dr. R. B. Allen of Clark University has been appointed professor of mathematics at Kenyon College, Gambier, Ohio.

Dr. H. F. Stecker has been promoted to an assistant professorship of mathematics at the Pennsylvania State College.

Professor W. B. Smith of Tulane University has been transferred from the chair of mathematics to the chair of philosophy.

Professor J. A. Miller of the University of Indiana has been appointed professor of mathematics and astronomy at Swarthmore College.

Dr. Edward Kasner has been promoted to an adjunct professorship of mathematics in Barnard College, Columbia University.

At McGill University A. S. Eve has been appointed assistant professor of mathematics, and Dr. H. T. Barnes associate professor of physics.

Professor D. E. Smith of Teachers' College, Columbia University, will spend the summer in Spain in seeking for early mathematical manuscripts and text books.

Professor Robert J. Aley of Indiana University was nominated for State superintendent of public instruction of Indiana at the recent Democratic State convention at Indianapolis.

The Randolph Jones Company of Chicago has prepared for sale a large number of Hanstein's model and goniostat for aid in teaching plane and solid geometry, perspective, shadow construction, etc.

Dr. Roxana Vivian, instructor in mathematics at Wellesly College, has obtained a years leave of absence, and will be a member of the faculty of the American College for girls in Constantinople the coming year.

THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. AUGUST-SEPTEMBER, 1906. NOS. 8-9.

SOLUTION OF A PROBLEM IN THE THEORY OF NUMBERS.

By E. B. ESCOTT, Chicago, Ill.

PROBLEM.* Let p^n-1 and n have a common divisor d. In what case is $p^{n/\delta}-1$ divisible by $d(p^{n/\delta}-1)$, δ being a divisor of d? This is always true for the particular value $\delta=1$.

Let $p^{n/d}=P$ and $d=\delta a$. Then the problem may be stated as follows: Given that $P^{\delta a}-1$ is divisible by δa, to find the necessary and sufficient conditions that the expression $\dfrac{P^a-1}{\delta a(P-1)}$ shall be integral. [When the latter is integral, it is evident that the condition that $P^{\delta a}-1$ shall be divisible by δa is *a fortiori* satisfied].

Let $a=a_1{}^a a_2{}^b\ldots$, where a_1, a_2, ... are distinct primes. First, suppose that a_1 is a factor of $P-1$. Then $P=1+a^m\kappa$, where $m \overline{>} 1$ and κ is any integer. Raising both members of this equation to the power a_1, we get

$$P^{a_1}=1+a_1\, a_1{}^m\kappa+\ldots\equiv 1(\text{mod } a_1{}^{m+1}),$$

from which we see that $P^{a_1}-1$ is divisible by $a_1{}^{m+1}$ and by no higher power of a_1. Similarly, $P^{a_1{}^a}-1\equiv 0$ (mod $a_1{}^{m+a}$), and therefore P^a-1 is divisible by $a_1{}^{m+a}$ and by no higher power of a_1.

*Question 2932, *L' Intermédiaire des Mathématiciens*, 13 (1906), p. 87, proposed by L. E. Dickson. For application of this question, see an article by L. E. Dickson, *On finite algebras*, *Goettinger Nachrichten*, July, 1905.

Secondly, suppose that α_2 is not a factor of $P-1$. Let $P^e-1\equiv 0$ (mod α_2), where e is the smallest exponent of P for which this congruence holds, and let P^e-1 be divisible by $\alpha_2{}^l$, but by no higher power of α_2. Then, since it is necessary that $P^{\alpha}-1\equiv 0$ (mod $\alpha_2{}^b$), if $l<b$, $P^{e\alpha_2{}^{b-l}}-1\equiv 0$ (mod $\alpha_2{}^b$), while this is the lowest number of this form which is divisible by $\alpha_2{}^b$. Therefore, it is necessary that α should be a multiple of $e\alpha_2{}^{b-l}$, and since e must be a factor of α_2-1 by Fermat's Theorem, and is therefore relatively prime to α_2, e must be a divisor of $\alpha/\alpha_2{}^b$. It is, therefore, necessary and sufficient that α be divisible by e.

If $l>b$, P^e-1 is the lowest number of this form which is divisible by $\alpha_2{}^b$, and we have as before that the necessary and sufficient condition is that α shall be divisible by e.

It can be shown that δ and $P-1$ can have no common factor. If they had a common prime factor δ_1, then since

$$\frac{P^{\alpha}-1}{P-1}=P^{\alpha-1}+P^{\alpha-2}+\ldots+P+1\equiv+1+\ldots+1\equiv\alpha\ (\text{mod}\ \delta_1),$$

δ_1 must be a factor of α. Suppose that $\delta_1=\alpha_1$. Then, since the numerator $P^{\alpha}-1$ is divisible by $\alpha_1{}^{m+a}$ and by no higher power, while in the denominator α contains $\alpha_1{}^a$ and $P-1$ contains $\alpha_1{}^m$, and δ contains α, to at least the first power, then the denominator would contain $\alpha_1{}^{m+a+1}$, so the fraction could not be equal to an integer.

Theorem. *The necessary and sufficient conditions that* $(P^{\alpha}-1)/\delta\alpha(P-1)$ *shall be integral are: (1)* α *must be divisible by* e*, the least integer such that* P^e-1 *is divisible by* α_κ *, where for* α_κ *is taken in turn the various prime factors of* α *not dividing* $P-1$*; (2)* δ *is any divisor of* $(P^{\alpha}-1)/\alpha(P-1)$.

EXAMPLE. Let $P=3^6$. Then $P-1=2^3.7.13$. Let $\alpha_1=7$, $\alpha_2=23$. Then $e=11$, since $P^{11}\equiv 1$ (mod 23). Therefore α must be a multiple of 7, 23, and 11. For example, let $\alpha=7.11.23$. Then δ may be taken as any divisor of $\dfrac{3^{6.7.11.23}-1}{7.11.23(3^6-1)}$; for example, $\delta=23$, 67, 547, 661, 1093, 3851, ..., or various numbers composed of these factors.

NOTE ON CERTAIN QUADRATIC NUMBER SYSTEMS FOR WHICH FACTORIZATION IS UNIQUE.

By G. B. BIRKHOFF.

If we define w either as a root of an equation

$$w^2=\frac{D}{4}=0\ldots(1),$$

where D is a negative integer $\equiv 0 \pmod 4$, or as a root of the equation

$$w^2+w+\frac{1-D}{4}=0\ldots(1'),$$

where D is a negative integer $\equiv 1 \pmod 4$, then the set of complex numbers

$$m+nw\ldots(2),$$

m and n real integers, possesses the notable property of being closed under addition and multiplication just as the set of real integers is.

The theorem concerning the uniqueness of the decomposition of real integers into prime factors is based on the algorithm of the greatest common divisor. This algorithm depends on the fact that for all real numbers ξ it is possible to choose a real integer m such that

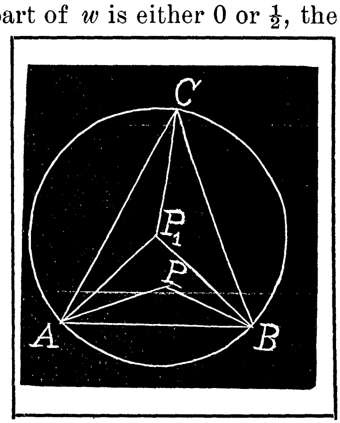
$$|\xi-m|<1\ldots(3).$$

The complete formal analogy between the complex number system (2) and the set of real integers leads to the remark: If w be such that for all complex numbers ξ it is possible to determine a complex number $m+nw$ such that

$$|\xi-(m+nw)|<1\ldots(4),$$

then the theorem of unique decomposition holds for these number systems. It is the purpose of this note to determine, by a geometrical scheme, for which of these systems condition (4) obtains.

In the complex plane we mark off the network of points $m+nw$, in particular the points $A\equiv 0$, $B\equiv 1$, $C\equiv w$. Since the real part of w is either 0 or $\frac{1}{2}$, the projection of C on AB falls on AB, which is of length 1, and *the triangle ABC is an acute or right triangle*. Further we write $w=r.e^{i\theta}$, whence noting that



$$|w|=\sqrt{-\frac{D}{4}} \text{ or } |w|\sqrt{\frac{1-D}{4}},$$

we obtain $\qquad r\geqq 1\ldots(5).$

Consider the points Q of the network $m+nw$ and a point P of the plane. At each point P there is clearly a least of the distances PQ; we call this distance $d(P)$. The condition (4) is then equivalent to the condition

$$d(P)<1\ldots(6) \text{ for all points } P.$$

In order to give the condition (5) explicit form we construct the circumcenter P_1 of the triangle ABC, so that

$$P_1A=P_1B=P_1C=\rho\ldots(7);$$

also since this triangle is acute or right, P_1 does not lie without the triangle. We first prove that

$$d(P_1)=\rho\ldots(8).$$

By (7) it is clear that if (8) is not true we have $PQ<\rho$ for some Q not A, B, or C; then Q is within the circumscribing circle. Assume if possible such a point Q to exist. From the character of the network it is clear that Q is not within the triangle ABC and accordingly Q must lie on one of the three circular segments, as AB. If then we imagine the network as made up of the intersection of two families of equidistant parallel lines, parallel to CA and CB, it is clear that the line parallel to CA through Q intersects AB produced from B, in β, and that the line parallel to CB through Q intersects AB, produced from A, in α. Thus

$$\angle \beta Q\alpha \geqq \angle BQA\ldots(9).$$

But

$$\angle BQA+\angle ACB>180^\circ\ldots(10),$$

since Q lies in the circle on the opposite side of AB from C. Further it is obvious that $\angle ACB=\angle \beta Q\alpha$. Adding corresponding sides of these equations we infer that, after dividing the inequality by 2, $\angle ACB>90^\circ$, which is not possible as the triangle ABC is an acute or right triangle. Hence (8) is proved.

Consider now an arbitrary point P of the triangle ABC, not P_1; then P lies on one of the triangles AP_1B, BP_1C, CP_1A, say AP_1B, so that $AP+PB<AP_1+P_1B(=2\rho)$. Therefore either AP or $PB<\rho$, whence

$$d(P)<\rho,\ P \text{ not at } P_1\ldots(11).$$

From (8) and (11) we infer that

$$\text{Maximum } d(P)=\rho\ldots(12)$$

for the triangle ABC, and hence for the plane, as the whole plane can be built of such triangles. Thus (6) is equivalent to

$$\rho<1\ldots(13).$$

But $\rho=\dfrac{BC}{2\sin\angle BAC}=\dfrac{\sqrt{(1+r^2-2r\cos\theta)}}{2\sin\theta}$, so that (13) is the same as $1+r^2-2r\cos\theta<2\sin\theta$, or $(r-\cos\theta)^2<3\sin^2\theta$. Noting that from (5), $r\geqq\cos\theta$, we find that

$$r<2\cos(\theta-60^\circ)\ldots(14).$$

This makes $r<2$ so that for systems (1)

$$-\frac{D}{4}=1,\ 2,\ 3,$$

and for systems (1′),

$$\frac{1-D}{4}=1,\ 2,\ 3,$$

are the possible cases. Investigating these cases using (14), we find that (4) obtains for the equations

$$w^2+1=0;\ w^2+2=0;\ w^2+w+1=0;\ w^2+w+2=0;\ w^2+w+3=0\ldots(15).$$

In these cases we have a greatest common divisor process.

If we define a prime number N of a system (1) or (1′) as one which has no factor n such that $1<|n|<|N|$, we have the theorem:

Every number N of one of the systems (15) can be decomposed essentially in but one way into prime factors. The non-essential variations are obtained frem the factors f such that $|f|=1$, which are the complete units.

The work here given completes the consideration of the case for a negative discriminant D, and agrees with the results of Dedekind, *Zahlentheorie*, fourth edition, page 450.

ON THE CHORD OF CONTACT OF TANGENTS TO A CONIC.

By W. D. LAMBERT, Washington, D. C.

If we wish to find the equation of the chord of contact of tangents drawn from a point $P_1\equiv(x_1, y_1)$, to the conic*

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0\ldots(1),$$

the straightforward way is to denote the unknown points of contact $P_2\equiv(x_2, y_2)$ and $P_3\equiv(x_3, y_3)$, and proceed to find the values of those coördinates. For this purpose we get two equations by substituting (x_2, y_2) and (x_3, y_3) in (1), and two more by expressing the fact that P_2 lies on the tangents through P_2 and P_3. These are

$$Ax_1x_2+\frac{B}{2}(x_1y_2+x_2y_1)+Cy_1y_2+\frac{D}{2}(x_1+x_2)+\frac{E}{2}(y_1+y_2)+F=0\ldots(2),$$

*I carry the work through for the general case, but as beginners are likely to be appalled by the mere length of an equation that is simple enough in principle, it is advisable in teaching to take at first a simpler special form of the equation of the conic.

$$Ax_1x_3+\frac{B}{2}(x_1y_3+x_3y_1)+Cy_1y_3+\frac{D}{2}(x_1+x_3)+\frac{E}{2}(y_1+y_3)+F=0\ldots(3).$$

These four equations give x_2, y_2, x_3, y_3 in terms of A, B, C, etc., and x_1, y_1. The required chord of contact is found by substituting their values in the formula for the straight line

$$\frac{y-y_2}{x-x_2}=\frac{y_3-y_2}{x_3-x_2}\ldots(4).$$

This direct solution is obvious in principle, but very tedious in the analytic work, and does not appear in text books. The favorite method in elementary treatises is to *assume* the result, namely, that the equation of the chord of contact of tangents to the conic (1) is

$$Ax_1x+\frac{B}{2}(xy_1+x_1y)+Cy_1y+\frac{D}{2}(x+x_1)+\frac{E}{2}(y+y_1)+F=0\ldots(5),$$

and then prove that (5) is a straight line passing through the points of contact P_2 and P_3. This process is certainly contrary to the spirit of *analytic* geometry, and seems obscure to many beginners.

The direct attack may be made easier by noting that we require not x_2, y_2, x_3, and y_3 themselves, but only certain combinations of them that are easy to find. Subtract (2) from (3) and factor so as to bring out x_3-x_2 and y_3-y_2. We may write the resulting equation in the form

$$\frac{y_3-y_2}{x_3-x_2}=-\frac{Ax_1+\frac{B}{2}y_1+\frac{D}{2}}{\frac{B}{2}x_1+Cy_1+\frac{E}{2}}.$$

Substitute this value in (4) and clear of fractions; after transposing the result may be written

$$Ax_1x+\frac{B}{2}(x_1y+xy_1)+Cy_1y+\frac{D}{2}x+\frac{E}{2}y-[Ax_1x_2+\frac{B}{2}(x_1y_2+x_2y_1)+Cy_1y_2]+$$

$$\frac{D}{2}x_2+\frac{E}{2}y_2=0\ldots(6).$$

Adding the identity (2) to (6), we get the required equation (5).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

265. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Obtain the reduced cubic $4\theta^3-I\theta+J=0$ of the biquadratic $ax^4+4bx^3+6cx^2+4dx+e=0$.

I. Solution by the PROPOSER.

Assume $ax^4+4bx^3+6cx^2+4dx+e\equiv a(x^2+2mx+p)(x^2+2nx+q)$.

$\therefore a(m+n)=2b$; $a(4mn+p+q)=6c$; $a(mq+np)=2d$; $apq=e$.

Let $amn=c-\theta$; then $a(p+q)=2c+4\theta$.

Substituting in the identity

$$\begin{vmatrix} 1 & 1 & 0 \\ m & n & 0 \\ p & q & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ n & m & 0 \\ q & p & 0 \end{vmatrix} \equiv \begin{vmatrix} 2 & m+n & p+q \\ m+n & 2mn & mq+np \\ p+q & mq+np & 2pq \end{vmatrix} \equiv 0$$

we obtain $\dfrac{8}{a^3}\begin{vmatrix} a & b & c+2\theta \\ b & c-\theta & d \\ c+2\theta & d & e \end{vmatrix}=0$; *i. e.*, $4\theta^3-I\theta+J=0$.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal.

$$ax^4+4bx^3+6cx^2+4dx+e\equiv x^4+\frac{4bx^3}{a}+\frac{6cx^2}{a}+\frac{4dx}{a}+\frac{e}{a}=0\ldots(1).$$

Let $x=y/a$; then (1) becomes $y^4+4by^3+6acy^2+4a^2dy+a^3e=0\ldots(2)$.

$y^4+4by^3+6acy^2+4a^2dy+a^3e+(ay+\beta)^2=(y^2+2by+\lambda)^2$, by Ferrari's solution; whence, by equating coefficients of like powers of y, and eliminating α and β, $\lambda^3-3ac\lambda^2+a^4(4bd-ae)\lambda-a^3(2d^2+2b^2e-3ace)=0$. Let $4^{\frac{1}{3}}\theta+ae=\lambda$, then

$$4\theta^3-4^{\frac{1}{3}}a^2(3c^2+ac-4bd)\theta+a^3[c(4bd-ac-2c^2)-(2cd^2+2b^2e-3ace)]=0\ldots(3),$$

which is of the required form.

266. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

Find the nth term and the sum of n terms of the series $1+3+7+17+\ldots$

I. Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let u_1, u_2, u_3, u_4, ..., u_n be the terms of the series, in the Calculus of Finite Differences, and let $\triangle u_1$, $\triangle^2 u_1$, $\triangle^3 u_1$, and $\triangle^4 u_1$ be the symbols for the different orders of differences.

The differences and the terms of this special series may be arranged as follows:

$$\begin{array}{llllllllll}
u_1= & 1 & & 3 & & 7 & & 17 & . & . & . \\
\triangle u_1= & . & 2 & & 4 & & 10 & . & . & . & . \\
\triangle^2 u_1= & . & . & 2 & & 6 & . & . & . & . & . \\
\triangle^3 u_1= & . & . & . & 4 & . & . & . & . & . & .
\end{array}$$

This shows that the third difference of u_1 or $\triangle^3 u_1$, is constant; and that, therefore, the fourth, and all higher differences, in this series, must vanish.

We have always, $u_1=u_1$; and then we have for the next term, $u_2=u_1+\triangle u_1=u_1(1+\triangle)$; and then, $u_3=u_1+\triangle u_1+\triangle(u_1+\triangle u_1)=u_1+\triangle u_1+\triangle u_1+\triangle^2 u_1=u_1+2\triangle u_1+\triangle^2 u_1=u_1(1+\triangle)^2$; and so on for higher orders; and in which $\triangle$ may be considered, first, as a symbol of operation, and second, as a symbol of quantity.

The symbols and operations may now be exhibited as follows, as they conform to the law of the Binomial Theorem:

$$\begin{aligned}
u_1&=u_1 \\
u_2&=u_1(1+\triangle) \\
u_3&=u_1(1+\triangle)^2 \\
u_4&=u_1(1+\triangle)^3 \\
&\cdots\cdots \\
u_n&=u_1(1+\triangle)^{n-1}
\end{aligned}$$

Expand the term for u_n and we have for its value:

$$u_n=u_1[1+(n-1)\triangle+\frac{(n-1)(n-2)}{2}\triangle^2+\frac{(n-1)(n-2)(n-3)}{2\times 3}\triangle^3+\ldots+].(A).$$

Let S_n=the value of the sum of n terms and we have:

$$S_n=u_1+u_2+u_3+u_4+\ldots+u_n.$$

We also have for the second member:

$$S_n=u_1[1+(1+\triangle)+(1+\triangle)^2+(1+\triangle)^3+(1+\triangle)^4+\ldots+(1+\triangle)^{n-1}]\ldots(B).$$

Sum (B) and we have:

$$S_n=\frac{u_1[(1+\triangle)^n-1]}{\triangle}.$$

Expand the term, $(1+\triangle)^n$, subtract 1, and divide by $\triangle$, and we have:

$$S_n=u_1[n+\frac{n(n-1)}{2}\triangle+\frac{n(n-1)(n-2)}{3!}\triangle^2+\frac{n(n-1)(n-2)(n-3)}{4!}\triangle^3$$
$$+\ldots]\ldots(C).$$

In (A) and (C) remove the brackets so as to unite the symbols of operation and the symbols of quantity and we have:

$$u_n=u_1+(n-1)\triangle u_1+\frac{(n-1)(n-2)}{2!}\triangle^2 u_1+\frac{(n-1)(n-2)(n-3)}{3!}\triangle^3 u_1+\ldots(D);$$

and

$$S_n=nu_1+\frac{n(n-1)}{2}\triangle u_1+\frac{n(n-1)(n-2)}{3!}\triangle^2 u_1$$
$$+\frac{n(n-1)(n-2)(n-3)}{4!}\triangle^3 u_1+\ldots(E).$$

In (D) and (E) substitute the values, $u_1=1$, $\triangle u_1=2$, $\triangle^2 u_1=2$, and $\triangle^3 u_1=4$, from the problem and its differences, and we have, after reduction:

$$u_n=\tfrac{1}{3}[2n^3-9n^2+19n-9]\ldots(F);\text{ and}$$
$$S_n=\tfrac{1}{6}[n^4-4n^3+11n^2-2n]\ldots(G).$$

Equations (F) and (G) are true for all values of n for the special series under consideration. When $n=4$, $u_n=u_4=17$, and $S_n=S_4=28$, as may be seen by inspecting the series in the problem.

But equations (D) and (E) are perfectly general when the series follows any regular law of progression; as we have to know, only, the value of the leading term, and the leading differences up to the difference that vanishes, to find the value of any term in a series and the sum of that series.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal., and G. W. GREENWOOD, M. A., Dunbar, Pa.

Let $S\equiv u_1+u_2x+u_3x^2+\ldots$ where $u_1=1$, $u_2=3$, $u_3=7$, and, in general, $u_n=2u_{n-1}+u_{n-2}$, x, of course, being less than unity, numerically.

$\therefore(1-2x-x^2)S=u_1+(u_2-2u_1)x$; *i. e.*,

$$S=\frac{1+x}{1-2x-x^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}$$

where $\alpha=1+\surd2$, $\beta=1-\surd2$, $A=\frac{1}{2}(1+\surd2)$, $B=\frac{1}{2}(1-\surd2)$.

$\therefore u_n=A\alpha^{n-1}+B\beta^{n-1}=\frac{1}{2}[(1+\surd2)^n+(1-\surd2)^n]$.

Let $S_n=u_1+u_2+\ldots+u_n$; $S_n(1-2-1)=u_1+u_2-2u_1-3u_n-u_{n-1}$; *i. e.*, $S_n=\frac{1}{2}[3u_n+u_{n-1}-2]$.

Solved in a similar manner by J. Scheffer.

CALCULUS.

219. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin mx\sin nx}{\sin x}dx$; (b) $\int_0^{\frac{1}{2}\pi}\frac{\cos mx\sin nx}{\sin x}dx$, where n is a positive integer. Also, modify the result for the case of m an integer.

Solution by S. A. COREY, Hiteman, Iowa.

When n is a positive integer, $\sin nx$ may be developed into a sine power series divisible by $\sin x$. Substituting this development in (a) or (b) each term may be readily integrated. This method is, of course, also applicable when m is an integer in (a), but when (b) m is an integer and n not an integer this method fails. In the latter case, as well as in the other cases, an approximate value of (a) and (b) may be deduced by the use of formula (1), page 12, AMERICAN MATHEMATICAL MONTHLY, January, 1906. By using no term higher than the third (the term involving B_2), and by obtaining $f'(x)$ and $f^{\mathrm{IV}}(x)$ by differentiating the right members of the following identities:

$$\text{(c)}\ \int\frac{\cos mx \sin nx}{\sin x}dx=\int\frac{\sin(m+n)x}{2\sin x}dx-\int\frac{\sin(m-n)x}{2\sin x}dx,$$

$$\text{(d)}\ \int\frac{\sin mx \sin nx}{\sin x}dx=\int\frac{\cos(m-n)x}{2\sin x}dx-\int\frac{\cos(m+n)x}{2\sin x}dx,$$

the following developments are obtained:

$$\text{(a)}=\frac{\pi}{2.2.r}\left\{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}+2\left[\frac{\sin(m\pi/2r)\ \sin(n\pi/2r)}{\sin(\pi/2r)}\right.\right.$$

$$\left.\left.+\frac{\sin(2m\pi/2r)\sin(2n\pi/2r)}{\sin(2\pi/2r)}+\ldots+\sin\frac{(r-1)m\pi}{2r}\sin\frac{(r-1)\ n\pi}{2r}\right]\right\}$$

$$-\frac{\pi^2}{6.2.2!(2r)^2}\left(s\sin\frac{s\pi}{2}-t\sin\frac{t\pi}{2}-2mn\right)$$

$$+\frac{\pi^4}{30.(2r)^4.2.4!}\left[(t^3-3t)\sin\frac{t\pi}{2}-(s^3-3s)\sin\frac{s\pi}{2}+2mn(m^2+n^2-1)\right]\ldots(1).$$

$$\text{(b)}=\frac{\pi}{2.2.r}\left\{\cos\frac{m\pi}{2}\sin\frac{n\pi}{2}+n+2\left[\frac{\cos(m\pi/2r)\ \sin(n\pi/2r)}{\sin(\pi/2r)}\right.\right.$$

$$\left.\left.+\frac{\cos(2m\pi/2r)\sin(2n\pi/2r)}{\sin(2\pi/2r)}+\ldots+\frac{\cos[(r-1)m\pi/2r]\ \sin[(r-1)n\pi/2r]}{\sin[(r-1)\pi/2r]}\right]\right\}$$

$$-\frac{\pi^2}{6.(2r)^2.2.2!}\left[s\cos\frac{s\pi}{2}-t\cos\frac{t\pi}{2}\right]$$

$$+\frac{\pi^4}{30.(2r)^4.2.4!}\left[(3s-s^3)\cos\frac{s\pi}{2}-(3t-t^3)\cos\right]\ldots(2),$$

where $s=(m+n)$, $t=(m-n)$. To insure rapid convergence, let $r>(m+n)$. If in (b), $(m+n)=(2p+1)$, m, n, and p integers, it is readily seen that the value of (b) is zero.

In order to test the accuracy of the work of computation as well as to test the convergence of the series, it is sometimes advisable to find the value of the definite integral with $r=2a$ after its value has been found with $r=a$. The work involved in this test is usually not great as the work that has been done when $r=a$ is made use of when $r=2a$.

To show the rapid convergence of (1) and (2) the two following simple examples will suffice:

$$\int_0^{\frac{1}{2}\pi}\frac{\cos(3x/2)\sin x}{\sin x}dx=\int_0^{\frac{1}{2}\pi}\cos\frac{3}{2}\frac{x}{}dx=\tfrac{2}{3}\sqrt{\tfrac{1}{2}}.\quad \text{Here } m=\tfrac{3}{2},\ n=1.\quad \text{Taking}$$

$r=3$, we have $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\cos(3x/2)\sin x}{\sin x}dx=\frac{\pi}{12}(1+\sqrt{\tfrac{1}{2}})+\frac{3\pi^2\sqrt{\frac{1}{2}}}{6^3.4}+\frac{7\pi^4\sqrt{\frac{1}{2}}}{30.6^4.2.4!}=.47141.$

Similarly, $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin(3x/2)\sin x}{\sin x}=\frac{\pi}{12}(2+3\sqrt{\tfrac{1}{2}})+\frac{3\pi^2}{6^3.4}(1+\sqrt{\tfrac{1}{2}})+\frac{\pi^4}{30.6^4.2.4!}$

$\times(\frac{27}{4})(1+\sqrt{\frac{1}{2}})=1.13807$, both results being correct to five decimal places.

Also solved by G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

137. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that all multiply perfect numbers of multiplicity n having only n distinct primes are comprised in $n=2, 3, 4$.

Solution by JACOB WESTLUND, Ph. D., Purdue University, Lafayette, Ind.

If $p_1, p_2, \ldots, p_n$ are the distinct prime factors of a number of multiplicity n, we must have $n<\overset{1,n}{\underset{i}{\Pi}}\dfrac{p_i}{p_i-1}$, and hence $n<\frac{2}{1}.\frac{3}{2}.\frac{5}{4}.\frac{7}{6}\ldots\dfrac{2n-1}{2n-2}$. But this is impossible when $n>4$, as seen by induction. For we have

$$(n+1)1.2.4.6\ldots2n=n.1.2.4.6\ldots2n+1.2.4.6\ldots2n.$$

Now if $n.1.2.4.6\ldots(2n-2)>2.3.5.7\ldots(2n-1)$, it follows that

$$(n+1)1.2.4.6\ldots2n>2.3.5.7\ldots(2n-1)2n+1.2.4.6\ldots2n, \text{ or}$$
$$(n+1)1.2.4.6\ldots2n>2.3.5.7\ldots(2n+1)-2.3.5.7\ldots(2n-1)+1.2.4.6\ldots2n.$$

Hence $(n+1)1.2.4.6\ldots2n>2.3.5.7\ldots(2n+1)$. For $n=5$ we have

$$5>\tfrac{2}{1}.\tfrac{3}{2}.\tfrac{5}{4}.\tfrac{7}{6}.\tfrac{9}{8}=\tfrac{315}{64}.$$

Hence for all values of $n>4$ we have $n>\overset{1,n}{\underset{i}{\Pi}}\dfrac{p_i}{p_i-1}$, which proves the theorem.

GEOMETRY.

293 (Incorrectly numbered 290). Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

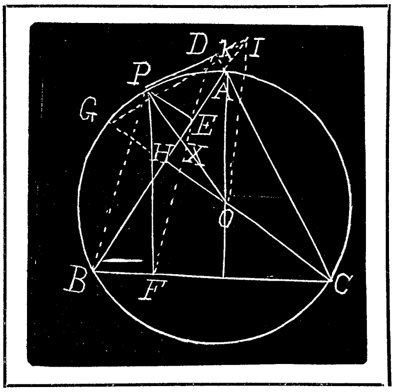
The pedal line of any point on a triangle's circum-circle bisects the distance between this point and the ortho-center of the triangle.

I. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Take, for convenience, an acute triangle ABC and the point P on the arc BC, the feet of the perpendiculars upon BC and CA being D and E, respectively. P is not supposed coincident with B or C. Call the orthocenter O, and *let* BO cut the circle again in G; *let* PQ intersect DE in H and AC in F. Basing the demonstration mainly upon pictorial evidence, we have, since the quadrilateral $PDEC$ is cyclic, $\angle PED=\angle PCD=\angle PGB=\angle GPE$. Hence the triangle PHE is isosceles, and therefore HEF is isosceles, and also $PH=HF$. It can easily be shown that OG is bisected by AC. Hence $\angle OFA=\angle GFA=\angle HFE=\angle HEF$. Hence DE bisects PF and is parallel to OF. It therefore bisects PO.

II. Solution by J. SCHEFFER. A. M., Hagerstown, Md.

Let P be a point in the circumference of the circle, and PD, PE, PF be the perpendiculars let fall from P upon the three sides of $\triangle ABC$, the *straight* line DEF is the pedal for the point P. Let O be the orthocenter, and draw OH perpendicular AB and extend it to G; join G with P and produce it until it meets the pedal DEF at K and the side AB produced at I; draw PB and IO. We now have $\angle PED=\angle PAD=\angle PBC=\angle PGH=\angle IPE$, PE being parallel to GH; hence $\angle KEI=\angle KIE$, being the complements of the equal angles IPE and PED; therefore $PK=KE=IK$. But $\angle HIO=\angle GIH=\angle DEI$. Therefore DX is parallel to IO, X being the point of intersection of PO and the pedal DEF. Since K is the middle point of PI, X must be the middle point of PO. Q. E. D.



294 (Incorrectly numbered 292). Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $(x^2+y^2)^3=mx^3$ to the problem of finding a cube m times a given cube.

[No solution has been received.]

295. Proposed by W. J. GREENSTREET, M. A., Editor Mathematical Gazette, Stroud, England.

A variable circle touches an ellipse, and the chord of contact through the other two points of intersection touches a similar coaxial ellipse. Find the locus of the center of the variable circle.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Let the circle be tangent at the point $P\equiv(a\cos\theta,\ b\sin\theta)$ to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let the chord of contact of the other two points touch at Q the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda^2.$$

Since the tangents at P and Q are equally inclined to the axes, the coördinates of Q are $(\lambda a\cos\theta,\ -\lambda b\sin\theta)$, or $(-\lambda a\cos\theta,\ \lambda b\sin\theta)$. Since a chord of a conic, tangent to a similar coaxial conic, is bisected at the point of contact, the center of the circle lies on the normal at Q; and also lies on the normal at P; *i. e.*, on

$$\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=\pm\lambda(a^2-b^2),\text{ and }\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

Hence the required locus is

$$\frac{4a^2x^2}{(1\pm\lambda)^2}+\frac{4b^2y^2}{(1\mp\lambda)^2}=a^2-b^2.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

270. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics in The University of Pennsylvania, Philadelphia, Pa.

Find the simplest integral form of the sum $y(y-1)\ldots(y-x)+2y(2y-1)\ldots(2y-x)+\ldots+zy(zy-1)\ldots(zy-x)$.*

271. Proposed by L. E. NEWCOMB, Los Gatos, California.

Sum the series $\dfrac{a}{b}+\dfrac{a^3}{3b^3}+\dfrac{a^5}{5b^5}+\ldots$ to ∞, $b>a$.

272. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the relations $x=\dfrac{ar+bs}{\lambda}=\dfrac{as-br}{\mu}=\dfrac{a\gamma-b\mu}{r}=\dfrac{a\mu+b\lambda}{s}$ between the finite real quantities $x,\ a,\ b,\ r,\ s,\ \lambda,\ \mu$ requires that $x^2=a^2+b^2$.

*This series is of frequent occurrence in certain investigations in Group Theory. Ed.

GEOMETRY.

299. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Show that the circle on any focal radius of an ellipse touches the auxiliary circle.

300. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Trisect an angle by means of a tractrix.

301. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $r=a(1+2\cos\theta)$ to the trisection of an angle. Describe the curve by continuous motion.

302. Proposed by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Through a given point within a circle draw any two chords, also a radius and a secant perpendicular to the radius. Let the extremities of the chords be taken as vertices of a quadrilateral. Show that the sides of the quadrilateral, produced when necessary, cut the secant in points equidistant, in pairs, from the given point. [A proof by Euclidean geometry is preferred, as the problem was originally given to a high school class.] Must the given point be within the circle?

CALCULUS.

228. Proposed by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A sphere, radius r, is dropped into a conical vessel whose vertex angle is 60°. Find the contents of the vessel between the vertex and the sphere by means of the formula, $V=\int\int\int dx\,dy\,dz$.

229. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Solve the differential equation $d^2y/dx^2=axy$.

MECHANICS.

192. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A solid sphere rolls down a trough formed by two planes which make with each other an angle 2α. Find, by the principle of *vis viva*, the expression for the time of rolling down the trough when the inclination of the trough to the horizon is β.

193. Proposed by W. J. GREENSTREET, M. A.. Editor of The Mathematical Gazette, Stroud, England.

Three light smoothly jointed rods stand like a tripod—the three edges of a regular tetrahedron. A rectangular board, weight w, stands on this like an easel. Find the thrust on the rod which does not touch the easel.

AVERAGE AND PROBABILITY.

179. Proposed by HENRY HEATON, Atlantic, Iowa.

Through every point of the circumference of a given circle, chords are drawn in every possible direction. What is their average length?

180. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

There are n numbers in a box numbered from 1 to n. A number is drawn and replaced n times. Show that on the average the number of repeats is $\left(\frac{n-1}{n}\right)^n n$.

GROUP THEORY.

15. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that there is no simple group of odd order less than 10,394.

A CORRECTION.

Professor G. Peano has called my attention to the fact that in my article on the transcendence of e and π, Vol. 12, page 223, line 9, of the MONTHLY, Newton's formulas which should be $a_n s_1 + a_{n-1} = 0$, $a_n s_2 + a_{n-1} s_1 + 2a_{n-2} = 0$, etc., are cited incorrectly. This makes it necessary to introduce in the function $\phi(x)$ at the bottom of page 221, the factor a_n^{np-1} and to modify the expressions on pages 222 and 223 in a corresponding way. The corrections apply to the proof of the transcendence of π, that for the transcendence of e requiring no change. The necessary modifications are to be found in Mr. Lennes' and my "Introduction to Infinitesimal Analysis" where the proof is reproduced. O. VEBLEN.

NOTES AND NEWS.

Mr. L. Weeks of Yale University has been appointed instructor in mathematics in Purdue University.

R. D. Carmichael has been appointed Professor of Mathematics in the Presbyterian College for Men at Anniston, Ala.

Prof. T. E. McKinney of Marietta College has accepted the Professorship of Mathematics in the Wesleyan University.

Dr. George Bruce Halsted has been appointed Professor of Mathematics in the State Normal School at Greeley, Colorado.

Mr. Paul Dorweiler and Mr. C. F. Hagenow have been appointed Instructors in Mathematics in Armour Institute of Technology.

Prof. Nelson L. Roray of the Utica (N. Y.) Free Academy, has been appointed Instructor in Mathematics in the Jersey City High School.

Dr. H. L. Coar, instructor in mathematics in the University of Illinois, has accepted the Professorship of Mathematics in Marietta College.

Dr. Sisam of the Annapolis Naval Academy, and Dr. Dodd of Iowa, have been appointed Instructors in Mathematics in the University of Illinois.

At Syracuse University, Mr. F. F. Decker has been advanced from Assistant in Mathematics to Instructor in Mathematics, and Mr. H. F. Hart was reappointed Assistant in Mathematics.

Dr. Glenn's connection with the MONTHLY ceased with the June-July number. We take this opportunity to extend to him our sincere thanks for the very efficient services rendered in conducting its affairs during the past year.

The June-July number of the MONTHLY was mailed the first week in September, the delay being caused by having to send proofs to Philadelphia. This issue was mailed three weeks late. We hope to have future issues mailed promptly on the 28th of each month, excepting July and August.

BOOKS.

On Finite Algebras. By Leonard Eugene Dickson. Presented to the Imperial Society of Science at Göttingen, by Herrn David Hilbert. Reprinted from the Bulletin of the Society. Pamphlet, 36 pages.

The author, in the introduction, states that the object of the article is the study of the independence of a set of postulates for a finite field of p^n elements, p being a prime, and cites as an illustration of his results two of the simpler finite algebras whose elements form a commutative group under addition, whose elements not equal to zero form a non-commutative group under multiplication, and which obey the left-hand, but not the right-hand, distributive law. To obtain an algebra whose elements form a commutative group under addition, with a law of multiplication which is commutative and distributive, but not associative, while division is always possible, he employs a linear algebra with coördinates in an arbitrary field F not having the modulus 2 and with units $1, i, j$ with a multiplication table in which $i\times i=j$, $i\times j=b+\beta i$, $j\times i=b+\beta i$, $j\times j=-\beta^2-8bi-2\beta j$, b and β being any marks of F such that $x^3-\beta x-b$ is irreducible in F.

The greater part of the investigation deals with the determination of all finite algebras of the two types illustrated, for each type of which the number of elements must be a power of a prime. The determination of all such algebras for n odd is made exhaustive, subject, however, to the validity of the following theorem in abstract group theory:

Any group of order p^n-1, p a prime and n an odd integer >1, contains a self-conjugate sub-group of order a power of a prime q, where q divides p^n-1, but not p^m-1, $m<n$.

This theorem the author verified through a very wide range of values. A demonstration of this theorem is desired. B. F. F.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. OCTOBER, 1906. No. 10

THE CONIC SECTIONS IN THE OLD JAPANESE MATHEMATICS.

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In a brief Note in the MONTHLY, Vol. XII, 1905, p. 166, it is remarked that "the most surprising fact about the old Japanese mathematics is that, while the most elementary parts were regarded as common property, the more advanced results were regarded as secrets which should be communicated to a very few." In fact, at most, one of the sons of one head master and two of his most highly versed disciples might be the keepers of the secrets of advanced mathematics as the successors of the head master. This curious fact, a parallel to which can be found in the Pythagorean school, interrupted the development of Japanese mathematics. But it became gradually impossible already before the Restoration of the Imperial government from the shogoon's authorities in 1868, to keep secret among a very few the proofs of theorems and the solutions of problems which they obtained from their predecessors or which they invented themselves.

Takakazu Seki (1642—1708) was the founder of the most famous school. This school was so flourishing that the mathematics of that school is the representative of Japanese mathematics and the development of mathematics in Japan was almost entirely accomplished by the scholars of that school. Seki, a contemporary of Newton and Leibnitz, invented the method of calculation called the Enri method, which well resembles the infinitesimal calculus. The now-living head master of the Seki school, Mr. Chōrin Kawakita, turned over to the library of the Tokyo Imperial University printed books and manuscript books belonging to him and containing theorems and problems kept in secret for about two hundred years. The number of these books together with those which the library bought reaches about two thousand, so that Prof. P. Harzer says that they are a

sufficient guarantee of the Japanese high esteem for mathematical knowledge. From this collection of Japanese mathematical works we may now see what were the secrets among the mathematicians in old Japan. I myself have some of the works of Japanese mathematics in my library. However, I may be allowed to say that I myself do not belong to the Seki school, or any other school of old Japanese mathematics.

In writing my work entitled "A brief history of the Japanese mathematics" in *Nieuw Archief voor Wiskunde*, 1904, pp. 296—324, and 1905, pp. 325—361, (the remaining part to be immediately published), I got most of the material from Mr. T. Endō's Dainihon-Sūgakushi or History of Mathematics in Great Japan, written in Japanese in 1896. Mr. Endō is one of the few men belonging to the old school to be found now-a-days, and his work is "written in a language not entirely intelligible and sometimes even repulsive to a student of modern mathematics, and in the characteristic tone peculiar to the men of the old school and altogether at variance with the spirit of modern mathematics," as one of my masters, Prof. Dr. R. Fujisawa of the Tokyo Imperial University, says in his paper, "Note on the mathematics of the old Japanese school" in the second international congress of mathematicians held at Paris in 1900 (see E. Duporcq's *compte rendu du congres*, pp. 379—393). So I compiled in my work the materials got from Mr. Endō's work in a manner easier to comprehend and quite different from that of Mr. Endō's work.

'I will take this opportunity to publish the content of the most famous work on the conic sections written about seventy years ago by a mathematician and containing curious and difficult problems of a quite different kind from those in European mathematics. But let me first give a history of the treatment of the conic sections in Japanese mathematics.

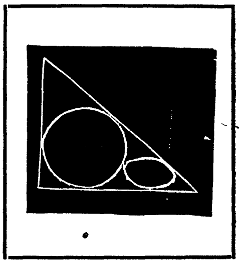
The mathematicians did not treat the sections of a right cone, as Apollonius did, but they treated the sections of a right cylinder, and therefore they knew only the ellipse until European mathematics was imported, and did not know the two curves, hyperbola and parabola, though they treated many other curves, algebraic or transcendental.

I do not know who first treated the ellipse. But we may say that Naomaru Ajima was the first mathematician who could completely calculate the length of an elliptic arc and the area of an elliptic segment, and consequently the whole perimeter and area of an ellipse, by a peculiar method which very well resembles the method of integration called by the mathematicians *Enri Method.* The literal meaning of the word Enri is "principle of a circle." Of all the problems that were investigated by the old Japanese mathematicians, the rectification and quadrature of a circle were the most attractive ones to them; so that several methods pertaining to the rectification, quadrature, and cubature of any curved line and any curved surface were brought together under the name of the Enri method, the ellipse being called *Sokuen*, considered as one kind of En or circles.

Naomaru Ajima quite reformed the hitherto prevailing method for the

rectification and quadrature of a circle, the new method much more resembling the infinitesimal calculus. He was born in 1739 and died in 1798. He became the head-master of the Seki school. It was one of his most remarkable achievements that the whole perimeter of an ellipse and the length of an elliptic arc were each expressed in an infinite series.

Ajima had written many manuscripts, but there was no printed one. His best disciple Makoto Kusaka (1764-1839) compiled in 1799 the manuscripts into a book which he named Fukyū-Sampō, which means a perpetual mathematics. This however, was not published as intended. The later scholars never failed to copy it and regarded it as one of the most esteemable works. It contains six problems relating to ellipses. The following enunciations will have something of interest, as indicating what problems the old mathematician solved. [The mathematicians of the old school did not enunicate problems minutely, but they drew always such a figure for every problem that we can understand the meaning of the problem; so they usually put the words "as in the figure" in the first part of the enunciation.]



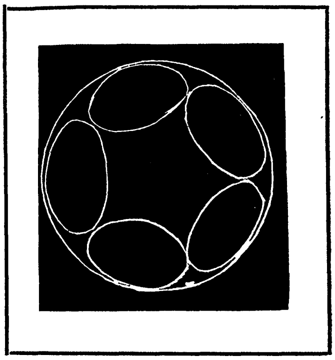
The 6th problem. As in the figure, a circle and an ellipse are inscribed in a right-angled triangle. Given the two sides of the triangle and the minor axes, find the major axes.

The 11th problem. An ellipse and a circle are inscribed in a rectangle, the major axis of the ellipse lying along the diagonal of the rectangle. Given the two sides of the rectangle and the minor axis, find the diameter of the circle.

The 20th problem. A spheroid is cut by a plane. Given the axes of the spheroid and the axes of the section, find the volume of one part of the spheroid.

The 21st problem. A spheroid is inscribed in a rectangular parallelopipedon. Given the three sides of the parallelopipedon and the minor axes of the spheroid, find the major axis.

The 22nd problem. In a given ellipse two equal chords and two equal circles are drawn tangent to the chords and the ellipse. Given the major and minor axes of the ellipse and diameter of the circles, find the length of the chords.



The 23rd problem. As in the figure, a certain number of equal ellipses are inscribed in a circle. Given the diameter of the circle and the major axis and the number of the ellipses, find the minor axis.

Kōhan Sakabe was another of the best pupils of Ajima. He died in 1824, the date of his birth being unknown. In 1810, Sakabe wrote Sampō-Tenzan-Shinan (a guide to the Tenzan method, which resembles the algebra.) One hundred and ninety-six typical problems are arranged in order

from easy problems to difficult ones. At the end of the work, we find the method of finding the length of the whole perimeter and of an arc of an ellipse which had been already discovered by his master Ajima. This was the first appearance of the problems pertaining to ellipses in printed books. It is said by some scholars that the part of the book relating to ellipses had been written by his pupil, Hisanori Kawai.

The successor of Kusaka was Yasushi Wada (or Nei Wada) who died in 1840, the date of birth being unknown. The Enri method was quite reformed by him. His improvement of the method was epoch-making in the history of Japanese mathematics, as Ajima's was. I will next explain his method of finding the perimeter of an ellipse, which is contained in his Enri-shinkō or Elementary lessons on the Enri method.

Divide the major axis $AB=a$ into $2n$ equal parts. Let PQ and $P'Q'$ be the mth and $m+1$th ordinates (called chō), the numbers being given to the ordinates from the minor axis $=b$, taken as the first ordinate to the right-hand ones and to the left-hand ones.

The $(m+1)$th Chōkaku, that is

$$m\text{th chō}-(m+1)\text{th chō}=\left(\frac{m}{n}\right)b \div \sqrt{1-\left(\frac{m}{n}\right)^2}.$$

The $(m+1)$th Haikaku, that is $2\times$elliptic arc intercepted between the mth and $(m+1)$th chō equals

$$\frac{a}{n}\sqrt{1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)} \div \sqrt{1-\left(\frac{m}{n}\right)^2}.$$

By the binomial expansion,

$$\left[1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)\right]^{\frac{1}{2}}=1-\tfrac{1}{2}\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)-\frac{1.1}{2.4}\left(\frac{m}{n}\right)^4\left(1-\frac{b^2}{a^2}\right)^2$$
$$-\frac{1.1.3}{2.4.6}\left(\frac{m}{n}\right)^6\left(1-\frac{b^2}{a^2}\right)^4-\frac{1.1.3.5}{2.4.6.8}\left(\frac{m}{n}\right)^8\left(1-\frac{b^2}{a^2}\right)^6-\dots$$

and

$$\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{1}{2}}=1+\tfrac{1}{2}\left(\frac{m}{n}\right)^2+\frac{1.3}{2.4}\left(\frac{m}{n}\right)^4+\frac{1.3.5}{2.4.6}\left(\frac{m}{n}\right)^6+\dots$$

By actual multiplication, $\left[1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)\right]^{\frac{1}{2}}\times\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{1}{2}}$ is expanded into a power series of $(m/n)^2$. Then $\frac{1}{2}\times$perimeter of the ellipse$=\lim\limits_{n=\infty}\sum^{m}$ (mth

$$\text{haikaku})=\pi a\left[1-\frac{1}{2}\cdot\frac{1}{2}\left(1-\frac{b^2}{a^2}\right)-\frac{1.1}{2.4}\cdot\frac{1.3}{2.4}\left(1-\frac{b^2}{a^2}\right)^2-\frac{1.1.3}{2.4.6}\cdot\frac{1.3.5}{2.4.6}\left(1-\frac{b^2}{a^2}\right)^3\cdots\right]$$

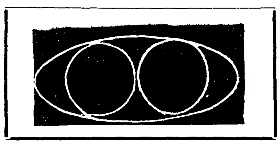
This was rewritten as follows:

$$\tfrac{1}{2}\times\text{perimeter}=\pi a\left\{1-\frac{1.1}{1^2}\cdot\frac{1-(b^2/a^2)}{4}-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\left[\frac{1-(b^2/a^2)}{4}\right]^2-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{3^2}\left[\frac{1-(b^2/a^2)}{4}\right]^3-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{3^2}\cdot\frac{5.7}{4^2}\left[\frac{1-(b^2/a^2)}{4}\right]^4-\cdots\right\}$$

In this series there appears the factor $1-(b^2/a^2)$, that is the square of eccentricity. This factor was called by the mathematician *ritsu* or modulus of the series.

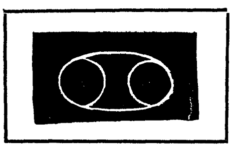
Kyō Uchida, well known by the name of Gokwan Uchida and living up to about 1877, was one of the best pupils of Kusaka and Wada. Kubota, a disciple of Uchida, wrote the first part of Enri-Shōhei-jutsu (the date of publication is not known). In 1855, the second part of the same work was written by Kumamoto, his fellow-student, and revised by Takemura of the same school. The first part is devoted to the methods of finding the center of gravity of plane figures and the second part to those of solid figures. Among the latter, the following problem is found: A spheroid is cut off by a plane parallel to one of its principal planes. Find the center of gravity of the remaining part.

Hiroshi (or Kwan) Hasegawa (1782—1832) was also one of the best pupils of Kusaka, but was expelled from Kusaka's school on account of his bad conduct, and was scorned by the mathematicians of the Seki school. Nevertheless he continued his study. He published many books by the names of his pupils. By the name of Tsunemitsu Murata, he published Sampō-Sokuen-Shōkai in 1831. Fifty problems relating to ellipses are arranged. This work was very famous and was respected by contemporary and later mathematicians. It is the most elegant work devoted to the study of ellipses. It will be interesting for us to know the content of this work, in order to learn the characteristic inclination of study of mathematics peculiar to the old Japanese school.



(1.) As in the figure, two equal circles in external contact are inscribed in an ellipse. Given the major and minor axes, fine the radius of the circles.

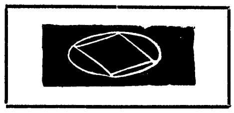
(2.) As in the figure, the circles in contact with an ellipse at the extremities of the major axis and having the maximum radius are drawn. Given the major and minor axes of the ellipse, find the maximum radius. [The circles are the osculating circles].



(3.) As in the figure, two equal circles not in contact are in-

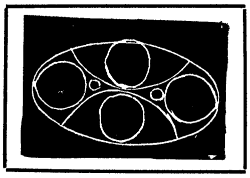
scribed in an ellipse. Given the major and minor axes and the radius of the circles, find the distance of the centers of the two circles.

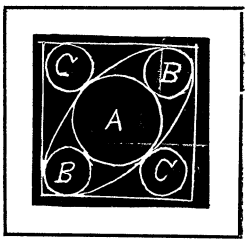
(4.) Two unequal circles in external contact are inscribed in an ellipse. Given the major and minor axes and the radius of one circle, find the radius of the other circle.



(5.) A rhombus is inscribed in an ellipse. Given the major and minor axes and the area of the rhombus, find the side of the rhombus.

(6.) As in the figure, two equal circular arcs are drawn and four equal large circles and two equal small circles are inscribed within an ellipse. Given the radius of the small circles, find the maximum of the radius of the large circles [two of the large circles are the osculating circles at the extremities of the major axis.]





(7.) As in the figure, an ellipse, one circle A, two equal circles B and two other equal circles C are drawn within a square. Given the radius of the circle C, find the maximum radius of the circle B [the circles B are the osculating circles at the extremities of the major axis.]

(8.) Four equal ellipses and five equal circles are drawn within a square, the extremities of the major axes of two neighboring ellipses being the same point. One of the equal circles has its center at the center of the square to which circle the four ellipses are tangent externally, the other four circles are in the areas formed by the intersection of pairs of ellipses. Given the major axis, find the minor axis.

(9.) Three equal ellipses and one circle are drawn within an equilateral triangle. The major axes of the ellipse coincide with the median of the triangle and the circle is tangent internally to the three ellipses. Given the major and minor axes of the ellipses, find the diameter of the circle.

(10.) As in the figure (23rd problem, p. 173), a certain number *e. g.* five, of equal ellipses are drawn within a circle. Given the diameter of the circle and the number of ellipses, find the major and minor axes.

(11.) As in the figure (23rd problem, p. 173), a certain number *e. g.* five, of equal ellipses are drawn within a circle). Given the major and minor axes, find the number of the ellipses when the diameter of the external circle is minimum.

(12.) Four equal circles are inscribed within the external curvelinear triangular spaces formed by two intersecting equal ellipses [having the same centre and the major axes at right angles]. Given the major axis, find the minor axis when the diameter of the circles is maximum.

(13.) An ellispe and a square are inscribed in a half isosceles trapezoid [the half part of an isosceles trapezoid divided by the join of the middle points of the two parallel sides]. Given the two parallel sides and the distance of the two parallel sides, find the side of the square.

(14.) Two equal chords not intersecting within the ellipse and two unequal circles are drawn within an ellipse, the circles being in internal contact with the ellipse at the extremities of the minor axis and tangent to the chords. Given the major and minor axes and the diameters of both circles, find the length of the chords.

(15.) The enunciation is the same as the preceeding; except that the chords intersect within an ellipse.

(16.) Two equal intersecting chords, two equal ellipses and two unequal circles are drawn within a circle, the circles and ellipses being tangent to the given circle and the intersecting chords. Given the diameter of the external circle and the major and minor axes, find the diameters of the two unequal circles.

(17.) Two circles A and B are intersecting, the center of B lying on the circumference of A; and two equal chords, two equal ellipses tangent to A and the two intersecting chords, and two equal circles C tangent to A, and one small circle D tangent to C and B and the intersecting chords are drawn, the major axes being parallel to the chords respectively. Given the major and minor axes, and the diameter of the circle A, find the diameter of the circle D.

(18.) An ellipse is inscribed in a circle and then a circle B is described on the minor axis of the ellipse. Two equal non-intersecting chords are drawn tangent to B and two equal circles D are inscribed in the smaller segments of the ellipse, cut off by the equal chords. Circles A and C are drawn tangent to the equal chords tangent externally to the ellipse and tangent internally to the given circle. Given the diameters of the given circle and the circle B, find the diameter of the circles D.

(19.) Two equal lines are drawn through the center of a rectangle. In the two pentagonal spaces two equal ellipses are drawn the major axis of which is equal and parallel to the shorter side of the rectangle. In the two triangular spaces, two equal circles A are drawn and two equal circles B are drawn tangent to the ellipses and the two intersecting lines. Given the two sides of the rectangle and the diameter of the circle A, find the diameter of the circle B.

(20.) Two equal ellipses are inscribed in a circle in such a way that three equal pentagons can be inscribed in the segments of the ellipse, a side of each pentagon lying in a common line and one pentagon being common to the two ellipses. Given the diameter of the circle, find the side of the pentagon and the major and minor axes of the ellipses.

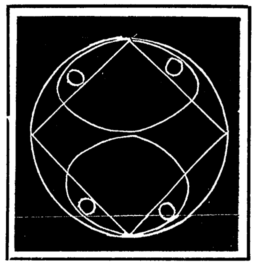
(21.) Two equal pentagons are inscribed in an ellipse in such a way that each of two diameters of the ellipse forms a side each of the two pentagons. Four equal circles are inscribed in the segments cut off by the sides of the pentagons. Given the major axis of the ellipse, find the diameter of the circles.

(22.) Six equal chords, two of which are parallel to the major axis, are drawn within an ellipse, and four equal circles are drawn in the four equal segments. Given the major and minor axes, find the diameter of the circles.

(23.) A chord is drawn in an ellipse and three unequal circles A, B, and

C are drawn, A and B each touching the ellipse in two points and tangent to the chord and C touching the ellipse in one point and tangent to the chord. Given the minor axis and the diameter of the circles A and B, find the diameter of the circle C.

(24.) Five unequal circles are drawn within an ellipse the circles touching the ellipse in two points, and the four chords A, B, C, D touching the two neighboring circles are drawn. Given the three chords A, B, C, find the fourth chord D.



(25.) As in the figure, a square, two equal ellipses and four equal circles are drawn within a circle. Given the diameter of the external circle, find the maximum diameter of the equal circles.

(26.) Two equal circular arcs and four equal small circles are inscribed in the segments of the ellipse cut off by these circular arcs are drawn within an ellipse. Given the major and minor axes, find the diameter of the small circles.

(27.) Two intersecting lines from the vertices of the acute angles and two equal ellipses having the major axes parallel to one side are drawn within a right-angled taiangle. Given the ratio of the major axis to the one side parallel to it, find the ratio of the minor axis to the other side parallel to it.

(28.) Two equal circles A with centers on the major axis, two equal circles B with centers on the minor axis, and one circle C whose center is the center of ellipse and tangent to circles A and B, are drawn within an ellipse. Given the major and minor axes, find the diameter of the circles A.

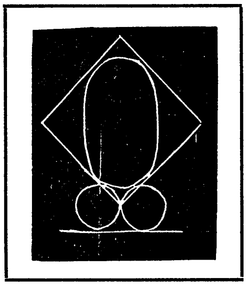
(29.) Circles whose centers are on major axis, and squares one of whose diagonals coincide with major axis are inscribed alternately within an ellipse. Given the major and minor axes, find the number of circles and the number of squares.

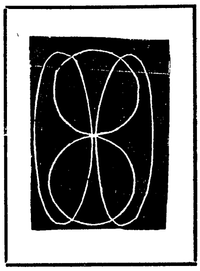
(30) and (31.) Two subsidiary problems of the preceeding.

(32.) An ellipse and a circle A are inscribed in a semi-circle, the major axis of the ellipse being parallel to the base of the semi-circle and the diameter of circle A coinciding with the minor axis of the ellipse. Two equal circles B tangent externally are inscribed in each of the crescents formed by the ellipse and circle A. Given the diameter of the small circles B, find the diameter of the semi-circle when the major axis is a maximum.

(33.) An ellipse is drawn within a square in such a way that it touches three sides of the square and its major axis is parallel to the side of the square. Four equal circles B whose centers form a square whose sides are parallel to the former square are inscribed in an ellipse, also a fifth equal circle B is discribed on the minor axis produced so that it is tangent to the ellipse and the side of the square. Two equal circles A are drawn tangent to this fifth circle B and two sides of the square. Given one side of the square, find the diameter of the circles A.

(34.) As in the figure, two equal circles, an ellipse and a square are drawn above a straight line, the ellipse touching the four sides of the the square and two circles. Given the diameter of the circles and the minor axis, find the major axis.

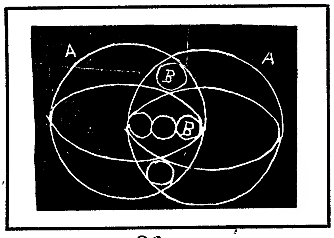




(35.) As in the figure, two equal ellipses and two equal circles are drawn. Given the major and minor axes, find the diameter of the circles.

(36.) As in the figure, two equal circles A and two equal ellipses and five equal small circles B are drawn. Given the diameter of the circles A, find the limiting length of the diameter of the small circles B.

(37.) As in the figure, two equal circles and one ellipse are drawn above a straight line, the major axis being parallel to the straight line. Given the diameter of the circles and the minor axis, find the major axis.



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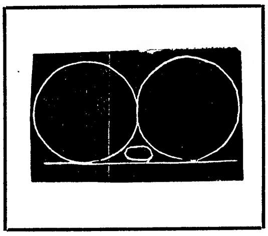

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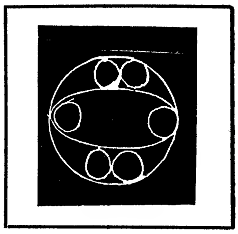

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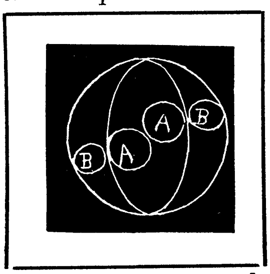
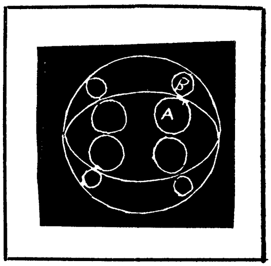
(38.) As in the figure, an ellipse and six equal circles are drawn within a circle. Given the diameter of the external circle, find the maximum diameter of the equal circles.

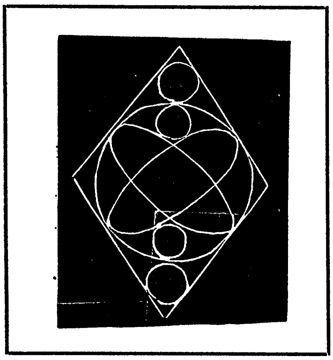


(39.) As in the figure, an ellipse, two equal circles A and two equal circles B are drawn within a circle. Given the diameters of the circles A and B, and the major axis, find the minor axis.



(40). As in the figure, an ellipse, four equal circles A and four equal circles B are drawn within a circle. Given the diameters of the circles A and B, and the major axis, find the minor axis.

(41.) As in the figure, a circle, two equal ellipses, two equal circles A and two equal circles B are drawn within a rhombus. Given the major and minor



axes, and the diameter of the circle A, find the diameter of the circles B.

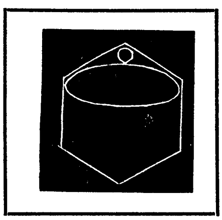
(42.) A spheroid and a certain number of equal small spheres surrounding the spheroid are drawn within a sphere. Given the major and minor axes, and the diameter of the small spheres, find the number of the small spheres.

(43.) Equal ellipses whose major axes coincide with diameters joining opposite vertices of a regular incribed polygon are inscribed within a circle. Equal circles are inscribed in the triangular space formed by two consecutive ellipses and the circle. Given the major and minor axes of the ellipses and the diameter of the small circles, to find the number of ellipses.

(44.) Equal ellipses whose major axes coincide with the diameters joining opposite vertices of a regular inscribed polygon are inscribed in the polygon. In the triangular spaces, formed at the vertices of each ellipse, by three consecutive ellipses, equal circles B are drawn, and in the space common to all the ellipses at the center of the polygon a circle A is drawn. Given the diameter of the external circle, find the diameter of the circle A when the diameter of the circles B is a maximum.

(45.) Two equal spheriods whose major axes are at right angles, and a certain number of equal spheres touching each of the spheroids and the sphere are drawn within a given sphere, the small spheres touching each other in pairs. Given the diameter of the external sphere and the number of the small spheres, find the diameter of the small spheres.

(46.) A circle A is concentric with a given circle. A certain number of equal ellipses tangent to the two circles at the extremities of the major axis are drawn. In the space common to two consecutive ellipses equal circles B are drawn. Given the diameter of the external circle, the major and minor axes, and the number of the ellipses, find the diameter of the circles B.



(47.) As in the figure, an ellipse and a circle are inscribed within a regular polygon. Given the major and minor axes, find the diameter of the circle.

(48.) Equal ellipses are drawn in a regular polygon. The number of ellipses is equal to the number of sides of the polygon; each ellipse is tangent to three consecutive sides of the polygon; and the major axis of each ellipse is parallel to the mid side of the three consecutive sides. In the areas common to two consecutive ellipses equal circles B are drawn. At the center of the polygon a circle A is drawn tangent to the ellipses. Given the side of the polygon, and the diameter of the circle A, find the diameter of the small circles.

(49.) Ellipses are incribed in a polygon as in problem 47. Given the major and minor axes, find the side of the polygon.

(50.) Equal ellipses are drawn within a polygon as in problem 49, and equal circles are drawn in the area common to two consecutive ellipses. Given the major and minor axes, find the diameter of the circles.*

As an appendix of this book, the author Murata described the minute demonstrations of the solutions of the sixth and eleventh problems of Ajima's Fukyū-Sampō in which the solutions of all problems were not demonstrated.

I will conclude this paper by adding that ellipsoids which were treated by Japanese mathematicians of the old school were spheroids only, called Chōryūen or elongated solid circle.

Tokyo, *December 1, 1905.*

ON THE FUNCTIONS WHICH HAVE A GIVEN ALGEBRAICAL ADDITION THEOREM.

By M. KABÀ, Tokyo, Japan.

Let $f(x)$ denote the uniform analytical function possessing a given algebraical addition theorem, that is,

$$f(x+y)=F[f(x), f(y)]\ldots(1).$$

It is evident that the function satisfies the equation

$$f(2x)=F[f(x), f(x)]\ldots(2).$$

In the first place we prove conversely that the uniform analytical function $\varphi(x)$ which satisfies equation (2) will satisfy equation (1). Set

$$\varphi(x+y)-F[\varphi(x),\ \varphi(y)]=F_1(x,\ y).$$

If we use $x+h$ instead of y and develop the function F_1 in a power series of h by Maclaurin's theorem, all the coefficients will be zero. In fact, as the function F is symmetrical in $\varphi(x)$ and $\varphi(y)$, we have the following:

$$\left[\frac{dF}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}F[\varphi(x),\ \varphi(x)],\quad \left[\frac{d^2F}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}F[\varphi(x),\ \varphi(x)],\ \ldots$$

And also we have the following:

*In justice to the author and to Editor Dickson, I wish to state that I changed the enunciation of all the problems unaccompanied by diagrams, to avoid making so many wood-cuts. Ed. F.

$$\left[\frac{d\,\varphi(2x+h)}{dh}\right]_{h=0}=\frac{1}{2}\frac{d\varphi(2x)}{dx},\quad \left[\frac{d^2\varphi(2x+h)}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2\varphi(2x)}{dx},\ \ldots$$

Therefore we have:

$$(F_1)_{h=0}=\varphi(2x)-F[\varphi(x),\ \varphi(x)],$$

$$\left[\frac{dF_1}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}\Big[\varphi(2x)-F[\varphi(x),\ \varphi(x)]\Big],$$

$$\left[\frac{d^2F_1}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}\Big[\varphi(2x)-F[\varphi(x),\ \varphi(x)]\Big],\ \ldots$$

But as $\varphi(x)$ satisfies the equation (2), we have:

$$(F_1)_{h=0}=0,\quad \left[\frac{dF_1}{dh}\right]_{h=0}=0,\quad \left[\frac{d^2F_1}{dh^2}\right]_{h=0}=0,\ \ldots$$

In the second place, we have to prove that there is a single uniform analytical function which satisfies the equation (2).

Let us suppose that the function $f(x)+\theta(x)$ satisfies the equation (2), where $f(x)$ satisfies (1) and therefore satisfies (2). From what we have said above, we know that $f(x)+\theta(x)$ possesses an addition theorem. But $f(x)$ possesses the given addition theorem (1). Therefore $f(x)+\theta(x)$ must be one of the three functions: Algebraical functions, simply periodical functions, doubly periodical functions. Hence $\theta(x)$ also must be one of these three functions, and therefore it possesses an addition theorem; that is

$$\theta(x+y)=F_1[\theta(x),\ \theta(y)],$$

and $\theta(x)$ has satisfied the eqnation

$$\theta(2x)=F_1[\theta(x),\ \theta(x)].$$

By the foregoing hypothesis, we have

$$f(2x)+\theta(2x)=F[f(x)+\theta(x),\ f(x)+\theta(x)],$$

that is,

$$F[f(x),f(x)]+F_1[\theta(x),\ \theta(x)]=F[f(x)+\theta(x),f(x)+\theta(x)].$$

Let us use y instead of $f(x)$ and z instead of $\theta(x)$, then we shall get the following identity:

$$F_1(z)=F(y+z)-F(y)$$

where F and F_1 are the algebraical functions of y and z. But as the first member of the identity is independent of y,

$$F'(y+z)-F'(y)=0.$$

Then $F'(y)$ must be a constant and $F(y)$ will have the form

$$F(y)=ay+b.$$

Also $f(x)$ and $\theta(x)$ will have the relations, respectively:

$$f(2x)=af(x)+b, \quad \theta(2x)=a\theta(x).$$

Therefore both $f(x)$ and $\theta(x)$ must be constants. Hence we have the following theorem:

The uniform analytical function satisfying a given addition theorem is the only one which satisfies the equation acquired by making $x=y$ on the addition theorem.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

The following note will be of value, since it calls attention to the fact that unless the law of a series is given, the *definite* determination of the series is impossible. If only a certain number of terms of a series is given and not the law of the series, the actual law of the series can in no case be really determined. All that can be done is to find the *simplest law* which the few given terms will obey. The solutions referred to may be justified on the ground that they were attempts to find this *simplest law*. ED. F.

NOTE ON THE SOLUTION OF PROBLEM 266, IN AUGUST-SEPTEMBER MONTHLY.

By CLARENCE E. COMSTOCK, Bradley Polytechnic Institute.

Find the nth term and the sum of n terms of the series $1+3+7+17+\ldots$

One writer assumes without the least excuse for such an assumption that it is an arithmetical series of the third order and proceeds by the method of finite differences, getting a result in accord with his assumption. Five terms of his series are $1+3+7+17+37+\ldots$

Another writer assumes with a little more plausible excuse that it is a recurring series of the second order and, of course, gets a result in accord with his assumption. Five terms of his series are $1+3+7+17+41+\ldots$

Suppose I make the assumption that it is an arithmetical series of the fourth order. I can then build up

$$1+3+7+17+38+76+\ldots \text{ or } 1+3+7+17+39+81+\ldots$$

or as many more as I care to take the time to construct, on the one supposition that it is an arithmetical series of the fourth order. Or let me assume it a recurring series of the fourth order, and get, using the scale of relation,

$$u_n=2u_{n-1}+u_{n-2}+0u_{n-3}+u_{n-4}, \text{ say, } 1+3+7+17+42+104+\ldots$$

The fact is, the problem is absolutely without meaning. There is nothing in the sequence $1+3+7+17+\ldots$ to show what the next term is. A series is not determined until its law is in some way stated or indicated.

267 Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Express the trigonometric functions of x as infinite continued fractions.

Solution by R. D. CARMICHAEL, Professor of Mathematics, Anniston, Ala.; and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let the series $u=A-B+C-D\ldots=\cfrac{a_1}{c_1+\cfrac{a_2}{c_2+\cfrac{a_3}{c_3+\ldots}}}$

To determine the a's we have the following (Euler's "Introductio," I, §367):

$$a_1=A.c_1,\ a_2=\frac{B.c_1c_2}{A-B},\ a_3=\frac{AC.c_2c_3}{(A-B)(B-C)},\ a_4=\frac{BD.c_3c_4}{(B-C)(C-D)}\ldots$$

in which the c's are to be so chosen that the a's are integral functions. Applying these formulas, we have

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\cfrac{x}{1+\cfrac{x^2}{2.3-x^2+\cfrac{2.3x^2}{4.5-x^2+\ldots}}}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\cfrac{1}{1+\cfrac{x^2}{1.2-x^2+\cfrac{1.2x^2}{3.4-x^2+\cfrac{3.4x^2}{5.6-x^2+\ldots}}}}$$

$$\tan x=\frac{2}{\pi-2x}-\frac{2}{\pi+2x}+\frac{2}{3\pi-2x}-\frac{2}{3\pi+2x}+\frac{2}{5\pi-2x}+\ldots^*$$

$$=\cfrac{2}{\pi-2x+\cfrac{4x}{(\pi-2x)^2+\cfrac{2\pi}{\pi^4-4x^2+\cfrac{(\pi-2x)(3\pi-2x)^2}{2x(\pi+2x)+\ldots}}}}$$

*Encyclopedia Britannica, Vol. XXIII, p. 572.

$$\cot x=\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}-\dots^{*}$$

$$=\cfrac{1}{x+\cfrac{x^2}{\pi-2x+\cfrac{(\pi-x)^2}{2x+\cfrac{(\pi+x)^2}{(\pi-2x)+\dots}}}}$$

$$\sec x=\frac{4\pi}{\pi^2-4x^2}-\frac{3.4\pi}{3^2\pi^2-4x^2}+\frac{4.5\pi}{5^2\pi^2-4x^2}-\frac{4.7\pi}{7^2\pi^2-4x^2}+\frac{4.9\pi}{9^2\pi^2-4x^2}-\dots\dagger$$

$$=\cfrac{4\pi}{\pi^2-4x^2+\cfrac{3(\pi^2-4x^2)^2}{6\pi^2+8x^2+\cfrac{(9\pi^2-4x^2)^2}{30\pi^2+8x^2+\dots}}}$$

$$\csc x=\frac{1}{x}+\frac{2x}{\pi^2-x^2}-\frac{2x}{2^2\pi^2-x^2}+\frac{2x}{3^2\pi^2-x^2}-\frac{2x}{4^2\pi^2-x^2}+\dots$$

$$=a-\frac{ax-1}{x}+\frac{2x}{\pi^2-x^2}-\frac{2x}{4\pi^2-x^2}+\dots\ddagger$$

$$=\cfrac{a}{1+\cfrac{ax-1}{1+\cfrac{2ax^3}{a\pi^2x-\pi^2-x^2-ax^3+\cfrac{2x(ax-2)(\pi^2-x^2)^2}{6\pi^2+\dots}}}}$$

Also solved by G. W. Greenwood.

268. Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Express the hyperbolic functions of x in the form of infinite continued fractions.

Solution by J. SCHEFFER, A. M., Professor of Mathematics, Kee Mar College, Hagerstown, Md.

For the conversion of $\tan x$ and $\tanh x$, Legendre has given a very elegant solution.

Putting $f(x)=1+\dfrac{a}{x}+\tfrac{1}{2}.\dfrac{a^2}{x(x+1)}+\dfrac{1}{2.3}.\dfrac{a^3}{x(x+1)(x+2)}+\dots$ we have

$$f(x+1)=1+\frac{a}{x+1}+\tfrac{1}{2}.\frac{a^2}{(x+1)(x+2)}+\frac{1}{2.3}.\frac{a^3}{(x+1)(x+2)(x+3)}+\dots$$

$$\therefore f(x)-f(x+1)=\frac{a}{x(x+1)}+\frac{a^2}{x(x+1)(x+2)}+\tfrac{1}{2}\frac{a^3}{x(x+1)(x+2)(x+3)}+\dots$$

**Encyclopedia Britannica*, Vol. XXIII, p. 572.

†Euler's *Institu. Calc. Diff.*, II, 8, No. 223—224.

‡The quantity a may have any integral value which will avoid negative terms in the continued fraction.

$$=\frac{a}{x(x+1)}\left[1+\frac{a}{x+2}+\tfrac{1}{2}\frac{a^2}{(x+2)(x+3)}+\dots\right]=\frac{a}{x(x+1)}f(x+2);$$

whence, putting $F(x)=\frac{a}{x}\cdot\frac{f(x+1)}{f(x)}$, $F(x+1)=\frac{a}{x+1}\,\frac{f(x+2)}{f(x+1)}$, we obtain

$$\frac{x}{a}\,\frac{f(x)}{f(x+1)}-\frac{x}{a}=\frac{1}{x+1}\cdot\frac{f(x+2)}{f(x+1)}.\quad \frac{1}{F(x)}-\frac{x}{a}=\frac{F(x+1)}{a},\quad F(x)=\frac{a}{x+F(x+1)}$$

$$\therefore F(x+1)=\frac{a}{x+1+F(x+1)},\quad F(x+2)=\frac{a}{x+2+F(x+2)},\ \text{etc.}$$

$$\therefore F(x)=\cfrac{a}{x+\cfrac{a}{x+1+\cfrac{a}{x+2+\cfrac{a}{x+3+\dots}}}},$$

$$\text{where } F(x)=\frac{a}{x}\,\frac{1+\dfrac{a}{x+1}+\tfrac{1}{2}\dfrac{a^2}{(x+1)(x+2)}+\dfrac{1}{2.3}\,\dfrac{a^3}{(x+1)(x+2)(x+3)}+\dots}{1+\dfrac{a}{x}+\tfrac{1}{2}\dfrac{a^2}{x(x+1)}+\dfrac{1}{2.3}\,\dfrac{a^3}{x(x+1)(x+2)}+\dots}$$

$$\text{For } x=\tfrac{1}{2},\ F(x)=2a.\frac{1+\dfrac{4a}{2.3}+\dfrac{16a^2}{2.3.4.5}+\dfrac{64a^3}{2.3.4.5.6.7}+\dots}{1+\dfrac{4a}{2}+\dfrac{16a^2}{2.3.4}+\dfrac{64a^3}{2.3.4.5.6}+\dots}=\frac{\epsilon^{2\sqrt{a}}-\epsilon^{-2\sqrt{a}}}{\epsilon^{2\sqrt{a}}+\epsilon^{-2\sqrt{a}}}\sqrt{a}.$$

$$\therefore \frac{e^{2\sqrt{a}}-e^{-2\sqrt{a}}}{e^{2\sqrt{a}}+e^{-2\sqrt{a}}}2\sqrt{a}=\cfrac{4a}{1+\cfrac{4a}{4+\cfrac{4a}{5+\dots}}}$$

$$\text{Putting } 2\sqrt{a}=x,\ \frac{e^x-e^{-x}}{e^x+e^{-x}}=\tanh x=\cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^3}{7+\dots}}}}$$

$$\text{Also } \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}=\cfrac{ix}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\dots}}}$$

$$\therefore i\tan x=\cfrac{ix}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{ix^3}{7-\dots}}}}\qquad \text{or, } \tan x=\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\dots}}}$$

a very simple and remarkable continued fraction.

Since $\sinh x = -i\sin ix$, and $\cosh x = \cos ix$, we get, by substituting ix for x in the continued fractions for $\sin x$ and $\cos x$,

$$\sinh x = \cfrac{x}{1 - \cfrac{x^3}{2.3 + x^2 - \cfrac{2.3x^2}{4.5 + x^2 - \cfrac{4.5x^2}{6.7 + x^2 - \text{etc.},}}}}$$

$$\cosh x = \cfrac{1}{1 - \cfrac{1}{2 + x^2 - \cfrac{2x^2}{3.4 + x^2 - \cfrac{3.4x^2}{5.6 + x^2 - \text{etc.}}}}}$$

GEOMETRY.

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289 (Incorrectly numbered 288). Proposed by C. N. SCHMALL, College of the City of New York.

From a point P on a given circle to draw two chords such that, (α) chord PA : chord $PB = m : n$ (a given ratio), and, (β) arc PA : arc $PB = 1 : 3$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the circle, radius r. Also let angle $POA = 2\theta$, angle $POB = 6\theta$. Then $PA = 2r\sin\theta$, $PB = 2r\sin 3\theta$.

$$\frac{PA}{PB} = \frac{\sin\theta}{\sin 3\theta} = \frac{1}{3-4\sin^2\theta} = \frac{m}{n}. \quad \therefore \sin\theta = \tfrac{1}{2}\sqrt{\frac{3m-n}{m}},\ \sin 3\theta = \frac{n}{2m}\sqrt{\frac{3m-n}{m}}.$$

$$\therefore \text{Make angle } POA = 2\sin^{-1}\tfrac{1}{2}\sqrt{\frac{3m-n}{m}};\ \text{angle } POB = 2\sin^{-1}\frac{n}{2m}\sqrt{\frac{3m-n}{m}}.$$

Then chord PA : chord $PB = m : n$; arc PA : arc $PB = 1 : 3$.

290 (Incorrectly numbered 289). Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

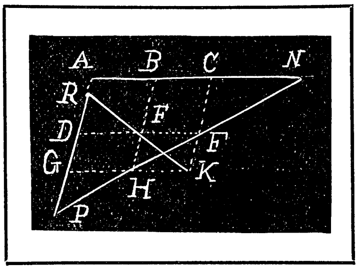
No solution has been received.

292 (Incorrectly numbered 290). Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given nine points lying by threes in three columns and in three rows, draw through them, by continuous motion, a broken line composed of only four straight segments, and passing but once through each of the nine points. [A current puzzle.]

Solution by G. I. HOPKINS, A. M., Professor of Mathematics and Astronomy, Manchester (N. H.) High School, and MISS IDA M. SCHOTTENFELS, A. M., New York, N. Y.

Let A, B, C, D, E, F, G, H, and K be the points. It is evident that A, C, K, and G will be the vertices of a parallelogram. Let BH be a median of this parallelogram, and E any point in the median except the center. Then the broken line $ANPRK$ will fulfill the conditions of the problem. This course fails if the middle row and middle column bisect each other. If the row DEF is not parallel to GK, then three lines, or a broken line of three segments will fulfill the conditions of the problem.



G. I. HOPKINS.

If BH and DF are not medians, take the course KER, RP, PN, NB. If BH and DF are medians, take the course KEA, AN, NP, PD; or KEA, AP, PN, NB.

IDA M. SCHOTTENFELS.

Also solved by the Proposer.

296 (Incorrectly numbered 294). Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

a) Suppose an indefinite line be pivoted at the end of a revolving radius whose center is the origin; and the initial position of the radius is coincident with the X-axis and the pivoted line perpendicular to it. As the radius revolves through equal amounts of arc the line moves to the right over corresponding equal intercepts on the X-axis. What is the equation of the locus of a point on the line whose distance from the end of the radius is equal to a diameter?

b) Show how the locus can be applied to the multisection of an angle.

c) Suppose the diameter be laid off in both directions.

No solution of this problem has been received.

297 (Incorrectly numbered 295). Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Md.

One side and the opposite angle of a triangle are fixed. Find the locus of the center of the inscribed circle. Solve by methods of analytic geometry.

I. Solution by C. N. SCHMALL, A. B., 89 Columbia Street, New York City.

This problem can be solved more easily and more neatly by Euclidean Geometry. Thus, referring to figure, we have,

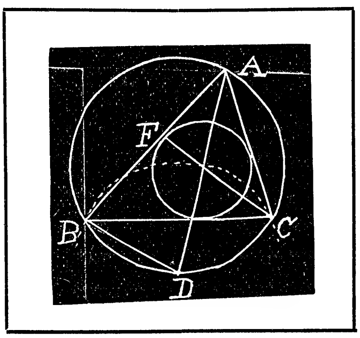
$\angle DCO = \angle DCB + \angle BCO = \angle DCB + \frac{1}{2}\angle BCA$,

$\angle DOC = \angle AOF = \angle OAC + \angle OCA$;

but $\angle OAC = \angle DCB$ (since arc $DC = DB$), and $\angle OCA = \frac{1}{2}\angle BCA$. Hence $\angle DCO = \angle DOC$.

Therefore $DC = DO$.

Hence, keeping BC constant and vertex A always on the arc BAC (making opposite angle constant) the locus of center O of the inscribed circle is a circle whose center is D and radius DC.



II. Solution by G. W. GREENWOOD, A. M., Dunbar, Pa.

Taking as x-axis the fixed side AB $(=2a)$ and its mid-point as origin, the in-center, since it lies on the bisectors of the angles A and B, must satisfy the equations $y=m_1(x-a)$, $y=m_2(x+a)$, where

$$m_1=\tan\left(\pi-\frac{A}{2}\right)=-\tan\frac{A}{2},\quad m_2=\tan\frac{B}{2}.$$

The opposite angle being constant, $A+B$ is constant. Hence

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\frac{-m_1+m_2}{1+m_1m_2}=\text{a constant}=c,\text{ say.}$$

Hence the in-center satisfies the equation $c(x^2+y^2-a^2)+2ay=0$.

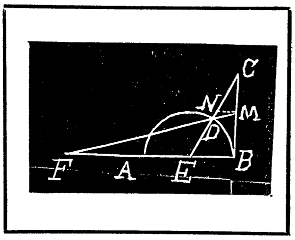
Also solved by J. Scheffer, William Hoover, and L. E. Newcomb.

298 (Incorrectly numbered 296). Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Given $AB=BC$ perpendicular to each other, and E and M their mid-points, respectively. On AB describe a semi-circle, and draw CE to meet the circumference in D. Draw DM cutting AB in F. In what ratio is AB divided by the point F?

Solution by C. N. SCHMALL. A. B., 89 Columbia Street, New York; L. E. NEWCOMB, Los Gatos, California; and A. H. HOLMES, Brunswick, Maine.

From the figure, constructed as described in the problem, we have



$FE : ED=NM : ND$. But $NM=\frac{1}{2}EB$, and

$ND=CD-CN=CD-\frac{1}{2}CE\ldots(1)$.
$\therefore FE : ED=\frac{1}{2}EB : CD-\frac{1}{2}CE\ldots(2)$.

Also $AB^2=CD.CD'$, where D' is the second intersection of the secant, CE, with the circle. Now let $FA=x$, $EB=r$, $CD=a$. Then $FE=x+r$, $CE=a+r$. Substituting these values in (1) and (2), we have

$$x+r : r=\tfrac{1}{2}r : a-\tfrac{1}{2}(a+r)\ldots(3),$$
$$(2r)^2=a(a+2r)\ldots(4).$$

From (4), $a=r(\pm\sqrt{5}-1)$, and from (3),

$$a=r+\frac{r^2}{x+r}.\quad \therefore r(\pm\sqrt{4}-1)=r+\frac{r^2}{x+r}.$$

Whence $x=(\pm\sqrt{5}+1)r$. $\therefore FA/AB=\frac{1}{2}(\pm\sqrt{5}+1)$.

Also solved by G. W. Greenwood and J. Scheffer.

CALCULUS.

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220 (Incorrectly numbered 219). For solution, see page 85.
221 (Incorrectly numbered 220, p. 90. For solution see page 148.

222 (Incorrectly numbered 221, p. 117). Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, O.

If a, b, c, ... represent all the prime numbers 2, 3, 5, ... prove that

$$(1+\frac{1}{a^2})(1+\frac{1}{b^2})(1+\frac{1}{c^2}).....=\frac{15}{\pi^2}.$$

No solution has been received.

223 (Incorrectly numbered 222, page 117). Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^1 (1+x^m)^n \log x\, dx$.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Assuming that n is integral, we have

$$\int_0^1 (1+x^m)^n \log x dx = \int \sum_{r=0}^{n} c_r x^{mr} \log x dx = \Sigma \frac{c_r[(mr+1)x^{mr+1}\log x - x^{mr+1}]}{(mr+1)^2}\Bigg]_0^1$$

$$=-\Sigma\frac{c_r}{(mr+1)^2}, \text{ since } \lim_{x\doteq 0} x^n \log x=0.$$

224 (Incorrectly numbered 221, page 153). Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Find $\lim_{x\doteq 0} \tan^{-1}x(\log x)$.

Solution by EDWIN L. RICH, Schenectady, N. Y.

$$\lim_{x\doteq 0} \tan^{-1}x\log x = \left.\frac{\log x}{\dfrac{1}{\tan^{-1}x}}\right\}_{x\doteq 0} = \left.\frac{\dfrac{d}{dx}(\log x)}{\dfrac{d}{dx}\left(\dfrac{1}{\tan^{-1}x}\right)}\right\}_{x\doteq 0}$$

$$=\frac{x^2+1}{x}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\text{ etc.}\right)^2\Bigg]_{x\doteq 0}$$

$$=(x^2+1)(1-\tfrac{1}{3}x^2+\tfrac{1}{5}x^4-\tfrac{1}{7}x^6+\text{ etc.})(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\text{ etc.})]_{x\doteq 0}=0.$$

Also solved by J. Scheffer and G. W. Greenwood.

225 (Incorrectly numbered 222). Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

If $s_n=2\left(\frac{1}{n}-\frac{2}{2n^3}+\frac{1}{5n^5}+\frac{1}{7n^7}-\frac{2}{9n^9}+\frac{1}{11n^{11}}+\ldots\right)$ prove that

$$\log 3=s_3+s_4,$$
$$\log 7=s_2+s_3+s_4,$$
$$\log 13=s_2+2s_3+s_4.$$

Solution by J. SCHEFFER, A. M., Professor of Mathematics, Kee Mar College, Hagerstown, Md.

We have $\log(1+\frac{1}{x})=\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-\frac{1}{4x^4}+\ldots$, and

$$\log(1-\frac{1}{x})=-\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{3x^3}-\frac{1}{4x^4}-\ldots$$

$$\therefore \tfrac{1}{2}\log\left(\frac{1+(1/x)}{1-(1/x)}\right)=\tfrac{1}{2}\log\left(\frac{x+1}{x-1}\right)=\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}+\frac{1}{9x^9}+\ldots$$

Likewise, $\tfrac{1}{2}\log\left(\frac{x^3+1}{x^3-1}\right)=\frac{1}{x^3}+\frac{1}{3x^9}+\frac{1}{5x^{15}}+\ldots$

$$\therefore \tfrac{1}{2}\log\left(\frac{x+1}{x-1}\right)-\tfrac{1}{2}\log\left(\frac{x^3+1}{x^3-1}\right)=\tfrac{1}{2}\log\left(\frac{x^2+x+1}{x^2-x+1}\right)=\frac{1}{x}-\frac{2}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}$$

$-\frac{1}{9x^9}+\ldots$ or $s_n=\log\left(\frac{n^2+n+1}{n^2-n+1}\right)$.

$\therefore s_3=\log\frac{13}{7}$ and $s_4=\log\frac{21}{13}$. $\therefore s_3+s_4=\log(\frac{13}{7}\times\frac{21}{13})=\log 3$. Likewise, $s_2+s_3+s_4=\log\frac{13}{3}+\log\frac{13}{7}+\log\frac{21}{13}=\log(\frac{13}{3}.\frac{13}{7}.\frac{21}{13})=\log 13$, and

$$s_2+2s_3+s_4=\log\tfrac{7}{3}+2\log\tfrac{13}{7}+\log\tfrac{21}{13}=\log(\tfrac{7}{3}.\tfrac{169}{49}.\tfrac{21}{13})=\log 13.$$

DIOPHANTINE ANALYSIS.

137 (Incorrectly numbered 136). Proposed by A. H. HOLMES, Brunswick, Maine.

Given $7x^2-111=y^2$. Required a value for y greater than unity which shall be a prime integer.

II. Solution by the PROPOSER.

$7x^2-111=y^2$. Put $x=u+a$. Then $7u^2+14au+7a^2-111=y^2=[pu+\sqrt{(7a^2-111)}]^2$; $7u^2+14au=p^2u^2+2pu\sqrt{(7a^2-111)}$.

$$\therefore u=\frac{14a-2p\sqrt{(7a^2-111)}}{P^2-7}.$$

Put $p=3$. $\therefore u=7a-3\sqrt{(7a^2-111)}$.

Put $a=5$. Then $u=35-24=11$, and $x=16$.

$7\times16^2-111=41^2$. $\therefore y=41$.

This solution shows that Professor Scheffer's solution in the June-July number, p. 148, did not yield the *least* prime, satisfying the equation, as is there stated. Ed. F.

AVERAGE AND PROBABILITY.

167. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

Solution by the PROPOSER.

Let the p segments taken at random in one instance be a, b, c, ... If p other points are taken at random on the line, the chance that the first will fall on a is a/l; that the second will fall on b is b/l; and so on. Hence, the chance that the p points will all fall on a, b, c, ... in a given order is $M(a.b.c...)/l^p$; and the chance that they will so fall in any order is $(1.2.3...p)M(a.b.c...)/l^p$.

The whole number of ways in which the $n+p-1$ points can be arranged is $(n+p-1)!$ The whole number of ways in which p points can fall on p segments (each in any order) is $(p!)(p!)$. We may then have

$$\frac{(1.2.3...p)\,M(a.b.c...)}{l^p}=\frac{(1.2.3...p)^2}{1.2.3...(n+p-1)}.$$

Hence, $M(a.b.c...)=\dfrac{(1.2.3...p)l^p}{1.2.3...(n+p-1)}$.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A circular arc, with center at one corner of a given square, is drawn through a point taken at random in the square. What is the average length of the arc within the square?

Solution by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the square, whose side is a, and let the coördinates of the random point P be (x, y), A being origin, AB the x axis and AD the y axis. Then the length of the arc through the random point is $2\phi\sqrt{(x^2+y^2)}$, where 2ϕ is the angle between the lines drawn from the origin to the intersection of the arc with the sides of the square. Passing to polar coördinates, we have $2\phi\rho$ for the length of the arc. But $\phi=\frac{1}{4}\pi-\sec^{-1}(\rho/a)$. Hence, the length of the arc is $2\rho[\frac{1}{4}\pi-\sec^{-1}(\rho/a)]=f(\rho)$, (say). Then the average length of the arc is

$$\Delta=\frac{f(\rho_1)+f(\rho_2)+\dots+f(\rho_n)}{n},\text{ where } n \text{ is the number of arcs,}$$

$$=\frac{\int_0^a 2\rho\frac{\pi}{4}d\rho+2\int_a^{\sqrt{2}a}\rho\left(\frac{\pi}{4}-\sec^{-1}\frac{\rho}{a}\right)d\rho}{\int_0^a d\rho+\int_a^{\sqrt{2}a}d\rho}$$

$$=\left[2\int_0^{\sqrt{2}a}\rho.\frac{\pi}{4}.d\rho-2\int_a^{\sqrt{2}a}\left(\sec^{-1}\frac{\rho}{a}\right)d\rho\right]/\sqrt{2}\,a=\tfrac{1}{2}\sqrt{2}a.$$

Solved in a similar manner with the same result by G. B. M. Zerr.

Solved in an entirely different manner with the result

$$\Delta=4a/3[\sqrt{2}-1-\int_0^{\frac{1}{4}\pi}\log(\tan\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)d\theta]$$

by Henry Heaton. Mr. Heaton assumes that the number of arcs of any given length within the square is proportional to the lengths of the arcs, and the whole number of arcs is equal to the number of points in the square. Now there is no reason why such an assumption may not be made, but such an assumption is certainly highly artificial. In our solution, it is clear that for every point we get a corresponding arc, and but one. If we take in all possible points in the square, we get all possible arcs. How to get all possible points is an open question, and an indefinite number of assumptions may be made as regards the distribution of the points when no law of distribution is given in the problem. The above solution tacitly assumes, (1) That the random points are distributed at equal angular distances on the arcs of circles, and (2) that the arcs of the circles cut the diagonal of the square at equal distances apart. By these assumptions every point in the square is considered. But either or both of these assumptions may be changed in any way at pleasure, each change giving different answers. Ed. F.

PROBLEMS FOR SOLUTION.

ALGEBRA.

273. Proposed by THEODORE L. DELAND, Treasury Department, Washington, D. C.

Three ingots of the precious metals were received at the Mint for assay, where it was found as follows: That in 3 grains of the first ingot and 2 grains of the second the gold was 3 times the silver; that in 2 grains of the first and 6 grains of the third the gold was 8 times the copper; that in 2 grains of the second and 3 grains of the third the silver was 5 times the copper; that in 1 grain of the first, 2 grains of the second, and 3 grains of the third the gold was 2 times the silver; that in 1 grain each of the first and second ingots there were 11 parts of gold to 5 parts of silver; and that 6 grains of the first, 5 grains of the second, and 2 grains of the third on being assayed proved to be 17 carats gold fine. There was no trace of any other metal in the ingots.

Required: The theoretical analysis of each of the three ingots.

274. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the limit of $\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\cdots$ where the squared numbers are the natural odd *primes* in order.

GEOMETRY.

303. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove that the pedal line of any point on a triangle's circum-circle bisects the distance from this point to the triangle's ortho-center.

304. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Find the tangent at the points $(a, 0)$ and $(0, a)$ to the locus $x^3+y^3=a^3$, and show that these points are points of inflection.

305. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

1. Suppose two radii R and R_1 revolve uniformly in the ratio 2 : 3. Find the equation of the locus of the intersection of R with the chord drawn from the end of the diameter to the extremity of R_1. 2. If the chord be drawn to the end of the diameter to the extremity of R, what is the locus of the intersection with R_1? 3. Show how an angle can be trisected by means of this curve.

CALCULUS.

230. Proposed by C. N. SCHMALL, College of the City of New York.

The greatest rectangle is inscribed in an ellipse, and the greatest ellipse in that rectangle, again the greatest rectangle in that (second) ellipse, and the greatest ellipse in that (second) rectangle, and so on *ad infinitum;* show that the sum of all the inscribed rectangles is equal to the area of the rectangle circumscribed about the given ellipse.

231. Proposed by EVA S. MAGLOTT, A. M., Professor of Mathematics, Ohio Northern University, Ada, O.

If a right circular cone stands on an ellipse, prove that the convex surface of the cone is $\frac{1}{2}\pi(OA+OA')(OA.OA')^{\frac{1}{2}}\sin\alpha$, where O is the vertex of ths cone, A and A' the extremities of the major axis of the ellipse, and α is the semi-angle of the cone at the vertex, using the formula $ds=\frac{1}{2}\rho\sqrt{(\rho^2+p^2)}d\theta$, where p is the perpendicular from the vertex to the base of the cone, ρ the distance from the foot of the perpendicular to any point in the perimeter of the base, and θ the angle between the major axis and ρ.

MECHANICS.

194. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

A body has a plane face resting on a rough wedge. The wedge is on a rough inclined plane, thick end down and thin edge horizontal. Find the condition that the body will slide down the wedge with constant acceleration, the wedge not slipping the while. Discuss the case in which the angle of friction for wedge and plane is greater than the angle of inclination of the plane.

195. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Particles slide from rest at the focus of a parabola, whose axis is vertical, down radius vectors, and are then allowed to move freely. Find the locus of the foci of their subsequent paths.

DIOPHANTINE ANALYSIS.

139. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

$2^{n-1}(2^n-1)$ is a multiply perfect number of multiplicity 2 when 2^n-1 is prime. Prove that there are no other multiply perfect numbers containing only 2 distinct primes.

140. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Determine (any way) whether the Diophantine equation $\left(\frac{2x-1}{3}\right)^3=x^2+y^2$ has any positive integer solutions.

141. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given that the highest factor of a prime p contained in $m!$ is p^{m-s}; find general expressions involving p and m and s, from which, when a solution is possible, m can be determined when s is a given integer and p is a given prime. Is it then possible in any case to have more solutions than one?

AVERAGE AND PROBABILITY.

181. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

At a sea-side excursion for x men there are boats enough for q men and carriages enough for z. But p do not care for driving, and q would feel indifferently comfortable on the water, while the rest do not care either way. Each man has what he prefers as long as a seat is left for him in carriages or boats, and those who do not care either way choose at random. Find the chance that all will be satisfied.

182. Proposed by L. MORDELL, Philadelphia, Pa.

Out of n straight lines whose lengths are 1, 2, 3, 4, ..., n inches, respectively, the number of ways in which 4 may be chosen which will form a quadrilateral in which a circle may be inscribed is $\frac{1}{48}[2n(n-2)(2n-5)-3+3(-1)^n]$.

MISCELLANEOUS.

163. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two straight streams of different volumes and velocities come together. Find the path of a body floating in mid-current of either.

NOTES AND NEWS.

Professor M. W. Haskell, of the University of California, has been promoted to a full professorship.

Professor H. P. Manning, of Brown University, has been promoted to an associate professorship of mathematics.

At the University of Rochester, Dr. A. S. Gale has been promoted to the Fayerweather Professorship of Mathematics.

Professor J. J. Quinn has charge of Manual Training and Mechanical Drawing in the Scottdale (Pa.) Public Schools.

Professors P. A. Lambert and A. E. Meake, of Lehigh University, have been promoted to full professorships of mathematics.

Milton L. Comstock, who for forty years was a teacher in Knox College, died at his home in Galesburg, Illinois, November 8, 1906, at the age of 82. Professor Comstock was elected to the chair of mathematics in Knox College in 1858 and held the position until 1898, when he was made Professor Emeritus. For years he was prominent in Teachers' Institute work in Illinois. He was a subscriber to the MONTHLY for several years.

BOOKS AND PERIODICALS.

A Laboratory Course in Physics for Secondary Schools. By Robert Andrews Millikan, Ph. D., Assistant in Physics in The University of Chicago, and Henry Gordon Gale, Ph. D., Instructor in Physics in The University of Chicago. 8vo. Flexible Cloth Cover, x+134 pages. Boston and Chicago: Ginn & Co.

This little laboratory manual designed for secondary schools contains 51 experiments covering a wide range of topics. The description of the experiments is adequate and the illustrations are good. The book will be welcomed by teachers of physics whose appropriation for apparatus is very limited, for much of the apparatus described can be made at home.

B. F. F.

Algebra for Secondary Schools. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Flexible Cloth Back, x+462 pages+50 pages of answers. Price, $1.20. Boston and Chicago: D. C. Heath & Co.

Among the distinctive features of this work are the use of formulas of physics and the introduction of problems involving the elementary laws of physics occurring at intervals throughout the book; graphical work occurring wherever the equation is introduced and given sufficient scope to acquaint the student with the great value of this work. Professor Wells has brought together in this book the best features of his other algebras.

B. F. F.

The Elements of Physics. By S. E. Coleman, A. B., A. M. (Harvard), Head of the Science Department and Teacher of Physics in the Oakland (Cal.) High School. 8vo. Cloth, 448 pages. Price, $1.25. Boston and Chicago: D. C. Heath & Co.

This book is well suited to High School use. The treatment of the general principles is ample, the selection of exercises is good, and the order of treatment of the various subjects is logical. B. F. F.

Text-Book in Algebra. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Half Leather Back, xi+561 pages. Price, $1.50. Boston and Chicago: D. C. Heath & Co.

This text-book contains over 4000 examples and problems, and a full treatment of all subjects of ordinary algebra. Free use is made of graphs. The book contains an Index in which are given references to all important operations and definitions. B. F. F.

Analytical Geometry. A First Course. By W. H. Maltbie, Woman's College of Baltimore, Md. 8vo. Paper Back, 142 pages. Published by the Author.

The aim of the author in preparing this book is to give a clear exposition of the methods of Analytical Geometry, leaving the working out of theorems largely to the student. In addition to the general treatment of the straight line and the conics, five appendices A, B, C, D, E, are added, in which are brief discussions of Infinities of Various Orders, Functionality, Permissible Operations, Projections, and Imaginaries. B. F. F.

Inductive Plane Geometry with Numerous Exercises, Theorems, and Problems for Advanced Work. By G. Irving Hopkins, Instructor in Mathematics and Astronomy, High School, Manchester, N. H. Revised Edition. Half Leather, vi+208 pages. Price, 75 cents. Boston and Chicago: D. C. Heath & Co.

Original work is the main feature of the method used by the author. "The early introduction of triangles, the minimum use of the method of superposition, numerical problems involving the use of the metric system of measures, and problems showing the applications of algebra to geometry" are special features of this edition.

On page 29 is given a direct proof of the theorem, *If the bisectors of the base angles of a triangle are equal, the triangle is isosceles.* This demonstration was published in Vol. IX, p. 43, of the MONTHLY, and it is believed was the first direct proof of the theorem ever published. B. F. F.

Practical Business Arithmetic. By John H. Moore, Commercial Department of the Charleston High School, Boston, and George W. Miner, Commercial Department, Westfield (Mass.) High School. 8vo. Cloth, viii+449 pages. Price, $1.00.

Some of the points of superiority claimed for this book are: (1) the emphasizing of fundamental operations; (2) freedom from useless definitions and theory; (3) problems dealing with live matter and correlated with the business activities of to-day; (4) graphic representations, scales, plots, and estimates given their proper share of attention; (5) and the illustration of a variety of business forms in attractive script.

The book is printed on a good quality of paper, is neatly bound, is very attractive in general appearance, and will prove of great value to all teachers of commercial arithmetic. B. F. F.

Text-Book of Mechanics. By Louis A. Martin, Jr. (M. E., Stevens; A. M., Columbia), Assistant Professor of Mathematics and Mechanics in Stevens Institute of Technology. Vol. I, Statics. First Edition. xii+142 pp. 8vo. Cloth. Price, $1.25. New York: John Wiley & Sons.

The author states in his preface that the work is based upon notes prepared for the use of Freshman and Sophomore classes at Stevens Institute of Technology. Analytical Geometry is used but no use is made of the Calculus. Numerous examples are introduced to illustrate each principle. B. F. F.

The Development of Symbolic Logic. A Critical-Historical Study of the Logical Calculus. By A. T. Shearman, M. A. 8vo. Cloth, x+242 pages. Price, 5 shillings. London: Williams and Norgate.

This book aims to show that there has been made during the last fifty years a definite advance in Symbolic Logic. The growth of the subject from the time when Boole wrote his *Laws of Thought* to the time when Russel, following for the most part the lines laid down by Peano, showed how to deal with a wider range of problems than those considered by Boole, has been traced. The book will be of great value to logicians and students of pure mathematics. B. F. F.

A First Course in Physics. By Robert Andrews Millikan, Ph. D., Assistant Professor of Physics in The University of Chicago, and Henry Gordon Gale, Ph. D., Instructor in Physics in The University of Chicago. 12mo. Cloth, viii +488 pages. Boston and Chicago: Ginn & Co.

This book, which is intended for a third year High School pupil, has grown out of the author's experience in developing the work in Physics in the School of Education of The University of Chicago. It emphasizes the historical and practical phases of the subject and closely connects it with the experience and observation of every day life. Human interest is added to the book by incorporating excellent portraits and brief biographies of such noted physicists as Maxwell, Faraday, Helmholtz, etc. A first class book for the High School. B. F. F.

Text-book on the Strength of Materials. By S. E. Slocum, Assistant Professor of Mathematics in the University of Illinois, and E. L. Hancock, Instructor in Applied Mechanics in Purdue University. 8vo. Cloth, xii+314 pages. Illustrated. List price, $2.00.

This book was produced with the aim of thoroughly representing the best theory and practice in an elementary form for the use of third year students in technical and engineering schools. It is divided into two parts; the first part containing 216 pages, presents the theoretical side of the subject, and the second part the experimental side. The entire subject is systematically developed, and in the treatment rigor and simplicity are happily combined. The book is very handsomely gotten up, the typography, illustrations, and mechanical execution being very good. B. F. F.

Essentials of Geometry, Plane and Solid. By Webster Wells. Half Leather, vii+402 pages. Price, $1.25. Boston and Chicago: D. C. Heath & Co.

In this book, the author recognizes the needs of the student, and meets them in such a way as to arouse his interest. The exercises, about 800, are carefully selected, the defintions are accurate, the demonstrations rigorous, and the discussions clear and logical.

B. F. F.

Space and Geometry in the Light of Physiological, Psychological, and Physical Inquiry. By Dr. Ernest Mach, Emeritus Professor in the University of Vienna. From the German by Thomas J. McCormack, Principal of the La Salle-Peru Township High School. 8vo. Cloth, 148 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

This little book is made up of three essays which were originally written for the *Monist* and which have appeared in that journal. The first essay treats "On Physiological, as Distinguished from Geometrical, Space"; the second, "On the Psychological and Natural Development of Geometry"; and the third, "Space and Geometry from the Point of View of Physical Inquiry." Dr. Mach's writings on a wide range of scientific subjects are well and favorably known to all readers interested in the progress of science. His familiarity with physiological, psychological, and physical theories has enabled him to discuss the subject from different points of view, and to throw light on the many divergent forms which the science of space has historically assumed. His discussion is thoroughly interesting and instructive. B. F. F.

Philolaus. By William Romaine Newbold, University of Pennsylvania. Reprinted from Archiv für Geschichte der Philosophie, XIX Band. Heft 2, 1905. Pamphlet, 176—217 pages.

The author attempts, in this article, to give an interpretation of several passages which were not satisfactorily explained by Boeckh in his monograph upon Philolaus, nor by any of his successors. The first is Philolaus' meaning of "Embodying" and "Splitting" ratios, a function which he applies to number. The second is the nature of the principles assumed by Philolaus and termed by him *περαι'νοντα* and *απειρα*. In the course of the author's discussion, he brings out a number of historical facts which will be of considerable interest to writers on the history of mathematics and astronomy.

A few typographical errors, and one error of statement on page 208, patent to the geometrician, exist. Here the author says, "Such a polygon [a polygon of fifty-eight sides] cannot even to-day be inscribed in a circle by any means known to mathematics." He, of course, means a regular polygon; and even then his statement is not true, for such a polygon is easily inscribed in a circle by means of a multisector, for example. By means of a straight edge and a pair of compasses, the problem is, of course, impossible. B. F. F.

Groups Generated by Two Operators which Transform Each Other into the Same Power. By Dr. G. A. Miller. Obilka z Prac Matematy czno-fizycznyck, T. XVII. Pamphlet, 4 pages.

The Value of Science. By M. H. Poincare. Translated by Dr. George Bruce Halsted. Reprinted from Popular Science Monthly. Pamphlet, 193—206 pages.

The Cattle Problem of Archimedes. By Professor Mansfield Merriman, Lehigh University. Reprinted from Popular Science Monthly. Pamphlet, 660—665 pages.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. NOVEMBER, 1906. No. 11

ON LINEAR ALGEBRAS.

By DR. L. E. DICKSON.

In the theory of algebraic numbers we consider such systems of numbers as the set of all numbers $r+si+ti^2$, in which r, s, t are rational, while i is an irrational number satisfying an equation $x^3-\beta x-b=0$ with rational coefficients. This set of numbers is called a field (or domain) since the sum, difference, product, or quotient, of any two of the numbers is likewise an unique number of the set. If we put $i^2=j$, we may consider this field to be a linear algebra composed of all numbers $r+si+tj$ (r, s, t rational), in which the "units" 1, i, j satisfy the relations

$$1^2=1,\ 1.i=i.1=i,\ 1.j=j.1=j, \tag{1}$$
$$i^2=j,\ i.j=j.i=b+\beta i,\ j^2=bi+\beta j. \tag{2}$$

In general, a linear triple algebra is determined by the "multiplication table" of its units [(1) and (2) in the above case] and by the field over which the coördinates r, s, t range [the field of rational numbers in the above case]. There is a very extensive literature on linear *associative* algebras, in which $i(jk)=(ij)k$ for any three units (not necessarily distinct). But only in rare instances, such as for quaternions, is division always possible; while then multiplication is not commutative. Another class of highly interesting linear algebras has been considered recently.* In these, multiplication is commutative and distributive, but not always associative, while *division is always uniquely possible*. We shall here

*Dickson, *Goettingen Nachrichten*, 1905, pp. 358—393; *Transactions American Mathematical Society*, Vol. 7 (1906), pp. 370—390; 514—522.

investigate such triple algebras by a new, simple method. Let* the units 1, i, j satisfy relations (1) and

(3) $$i^2=j,\ ij=ji=b+\beta i,\ j^2=d+\delta i+Dj,$$

$x^3-\beta x-b$ being irreducible in the reference field F. Then

(4) $$(r+si+tj)(x+yi+zj)=P+Qi+Rj,$$

where P, Q, R are given (by detached coefficients) by

(5) $$\begin{array}{r|ccc} & x & y & z \\ \hline P= & r & tb & sb+td \\ Q= & s & r+t\beta & s\beta+t\delta \\ R= & t & s & r+tD \end{array}$$

The determinant (of the third order) of this array equals

(6) $$\begin{aligned}\triangle =r^3+r^2t(\beta+D)+rt^2(\beta D-d)-rs^2\beta-rst(\delta+2b)\\ +s^3b+s^2td-st^2bD+t^3(b\delta-d\beta).\end{aligned}$$

Uniqueness of division requires that, for given values of r, s, t (not all zero) and P, Q, R, there shall exist unique solutions x, y, z of (4) and hence of (5). *Hence division is always uniquely possible if and only if* $\triangle=0$ *implies* $r=s=t=0$. We thus require that the Hessian H of the cubic form $\triangle$ shall differ from $\triangle$ only by a constant factor. By definition, this Hessian is

$$H=\begin{vmatrix} \triangle_{rr} & \triangle_{rs} & \triangle_{rt} \\ \triangle_{rs} & \triangle_{ss} & \triangle_{st} \\ \triangle_{rt} & \triangle_{st} & \triangle_{tt} \end{vmatrix},\quad \triangle_{rr}\equiv\frac{\partial^2\triangle}{\partial r^2},\ \triangle_{rs}\equiv\frac{\partial^2\triangle}{\partial r\,\partial s},\ \text{etc.}$$

For the cubic form $\triangle$, given by (6),

$$\begin{aligned}&\triangle_{rr}=6r+2t(\beta+D),\quad \triangle_{rs}=-2s\beta-t(\delta+2b),\quad \triangle_{rt}=2r(\beta+D)+2t(\beta D-d)\\ &\qquad\qquad -s(\delta+2b),\\ &\triangle_{ss}=-2r\beta+6sb+2td,\quad \triangle_{st}=-r(\delta+2b)+2sd-2tbD,\quad \triangle_{tt}=2r(\beta D-d)-2sbD\\ &\qquad\qquad +6t(b\delta-d\beta).\end{aligned}$$

The cubic form H is found to have the coefficients

$$\begin{aligned}&r^3\ :\ A=8\beta(\beta+D)^2-6(\delta+2b)^2-24\beta(\beta D-d),\\ &s^3\ :\ B=8\beta^2bD+8\beta d(\delta+2b)-6b(\delta+2b)^2,\\ &t^3\ :\ C=[24d(\beta+D)-6(\delta+2b)^2]\,(b\delta-d\beta)+8bD(\delta+2b)(\beta D-d)-8b^2D^2(\beta+D)\\ &\qquad -8d(\beta D-d)^2,\end{aligned}$$

*It is easily shown that there must occur a "modulus" 1, satisfying (1), and that not every element e satisfies a quadratic equation. Hence there exists an element i whose square is linearly independent of 1, i; it may be taken as the third unit j.

r^2s : $J=24b\beta D+72b(\beta D-d)+24d(\delta+2b)-24b(\beta+D)^2$,
s^2t : $K=2d(\delta+2b)^2+[24b(\delta+2b)-16\beta d](\beta D-d)-24\beta^2(d\delta-d\beta)$
$\quad -(24b^2D+8d^2)(\beta+D)$,
rs^2 : $L=-72b^2D-24d^2-16\beta d(\beta+D)-8\beta^2(\beta D-d)-2\beta(\delta+2b)^2$
$\quad +24b(\beta+D)(\delta+2b)$,
r^2t : $M=[24d+8\beta(\beta+D)](\beta D-d)-72\beta(b\delta-d\beta)-24bD(\delta+2b)$
$\quad +2(\beta+d)(\delta+2b)^2-8d(\beta+D)^2$,

the coefficients of rt^2, st^2 and rst not being exhibited.

We require that the coefficients A, B, ... of the Hessian shall be proportional to the corresponding coefficients of $\triangle$. It will not be necessary to examine all of these conditions. We exclude at present the special case in which the reference field F has a modulus* 2 or 3, so that we may divide by these numbers. Since there is no term r^2s in $\triangle$, we have $J=0$,

(7) $$2b\beta D-b\beta^2-bD^2-bd+d\delta=0.$$

The condition $A:B=1:b$ is then satisfied. From $C:A=(b\delta-d\beta):1$,

(8) $$\beta^4d-\beta^3b\delta-\beta^3dD+\beta^2b\delta D-\beta Dd^2+\beta b^2D^2+2\delta dDb-2b^2dD-b^2D^3-d^3=0.$$

From $K:A=d:1$,

(9) $$[d\beta+3b(\delta+2b)](\beta D-d)+d(\delta+2b)^2-\beta d(\beta+D)^2-3\beta^2(b\delta-d\beta) -(d^2+3b^2D)(\beta+D)=0.$$

Now $b\neq 0$ in view of the assumed irreducibility of

(10) $$x^3-\beta x-b.$$

We assume here that $\beta\neq 0$. We may set

(11) $$\delta=xb,\ D=yb,\ d=z\beta^2.$$

Then (7) has the factor $b\beta^2$, and we get

(12) $$xz-z=(y-1)^2.$$

Next, we multiply (8) and (9) by d, and eliminate $d\delta$ by means of (7). In the resulting conditions make the substitution (11). After suitable factorizations, we get

(13) $$\beta^5(z+1)G=0,\ \beta^4(y+2)G=0,$$
where $$G=b^2(y-1)^3-\beta^3z^2(z+y-1).$$

*The reader may hold F to be an algebraic field; it is only for completeness that modular fields are included (with proper restrictions).

If $G=0$, set $Y=b(y-1)$, $Z=\beta z$. Then $bG=0$ becomes

$$Y^3-\beta YZ^2-bZ^3=0.$$

But (10) is irreducible. Hence $Y=Z=0$, so that $y=1, z=0$. Thus $D=\beta$, $d=0$. Now $L:A=-\beta:1$. Thus

$$0=L+\beta A=-8\beta(\delta-b)^2.$$

Hence $\delta=b$. Hence relations (3) reduce to (2), so that *the algebra is a field.*

Let next $G\neq 0$. Then $z=-1$, $y=-2$, $x=-8$, by (13) and (12). Thus $\delta=-8b$, $D=-2\beta$, $d=-\beta^2$ by (11). The remaining conditions are seen to be satisfied. Indeed, the Hessian now equals

$$(32\beta^3-216b^2)\triangle.$$

The multiplication table of the resulting algebra is given by (1) and

$$i^2=j,\quad ij=ji=b+\beta i,\quad j^2=-\beta^2-8bi-2\beta j. \tag{14}$$

A direct proof that division is always uniquely possible in this remarkable algebra is given in the papers cited above.

Consider next the case $\beta=0$. Eliminating $d\delta$ between (7) and (8), we get $b^2D^3-d^3=0$. If $d\neq 0$, set $x=bD/d$. Then $x^3-b=0$, contrary to the irreducibility of (10). Hence $d=0$, $D=0$. Set $\lambda=-(\delta+2b)$. Then

$$\triangle=r^3+rst\lambda+s^3b+t^3b\delta,\quad H=-6r^3\lambda^2-6t^3\lambda^2b\delta-6s^3\lambda^2b+rst(2\lambda^3+6^3b^2\delta).$$

Hence H is a constant times $\triangle$ if and only if

$$2\lambda^3+6^3b^2\delta=-6\lambda^3,$$

viz., if $(\delta-b)^2(\delta+8b)=0$. The resulting algebras are again (2) and (14).

For completeness, we treat the excluded case* of a field with modulus 3. Let first $\beta\neq 0$. From $B:A=b:1$,

$$b(\beta-D)^2=d(\delta+2b). \tag{15}$$

Reducing (9) modulo 3 and eliminating $\delta+2b$ by (15), we get

$$(\beta-D)[d^2-\beta d^2(\beta-D)+b^2(\beta-D)^3]=0.$$

If the second factor vanished, (10) would have a rational root

$$x=-b(\beta-D)/d.$$

*For modulus 2, $H\equiv 0$. A discussion shows that the algebra is a field.

Hence $\beta=D$. Hence $A=-\beta^3$, $L=\beta^4+\beta(\delta+2b)^2$. Hence $L:A=-\beta:1$ gives $\delta\equiv b \pmod 3$. Then (8) becomes $d^2(d+\beta^2)\equiv 0$. Hence $d=0$ or $-\beta^2$, giving again algebras (2) and (14). Finally, let $\beta=0$. Then $A\equiv 0$, so that the Hessian must be identically zero. From C, K, we get

$$bdD(\delta+2b)+b^2D^3+d^3=0,\quad d^2D=d(\delta+2b)^2.$$

If $d\neq 0$, we eliminate $\delta+2b$ and get $(b^2D^3-d^3)^2=0$. Hence $(bDd^{-1})^3-b=0$, contrary to the irreducibility of (10). Hence $d=0$, $D=0$. Then $H=(\delta+2b)^3rst$, so that $\delta\equiv b$. The algebra is again the field algebra.

In summary, *the only algebras with the required properties are the obvious field-algebra and the remarkable non-field algebra* (14).

The University of Chicago, *September, 1906.*

THE DIVISION OF ANGLES INTO n EQUAL PARTS.

By J. SAMSONOFF, New York City.

Principle: If by the method given later one has constructed the geometrical loci of vertices of triangles with base XA, whose remote angle at the base is equal respectively to one, two, three, ..., $(n-1)$ times the angle at the vertex of the respective triangle, any angle CAY is geometrically divisible by 2, 3, ..., n.

Given: Base XA and curves a, b, c, d, ..., the geometrical loci of vertices of triangles, where the remote

$$\angle aXA=\angle XaA,\ \angle bXA=2\angle XbA,\ \ldots,$$
$$\angle eXA=(n-1)\angle XeA.$$

To prove: $\angle CAY$ is geometrically divisible into 2, 3, 4, ..., n equal parts.

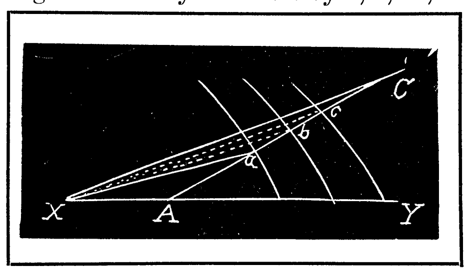


Fig. 1.

Proof: (1) In $\triangle XaA$, $\angle aXA=\angle XaA$ (by hypothesis), $\angle CAY=\angle aXA+\angle XaA$ (the exterior angle of $\triangle aXA$). Therefore the measure of $\angle XaA$ must be equal to $\frac{1}{2}$ of the measure of $\angle CAY$.

(2) In $\triangle XbA$, $\angle bXA=2\angle XbA$ (by hypothesis), $\angle CAY=\angle bXA+\angle XbA$ (the exterior angle of $\triangle XbA$). Therefore the measure of $\angle XbA$ must be equal to $\frac{1}{3}$ of the measure of $\angle CAY$.

(n) In $\triangle XeA$, $\angle eXA=(n-1)\angle XeA$ (by hypothesis), $\angle CAY=\angle eXA+\angle XeA$ (the exterior angle of $\triangle eXA$). Therefore the measure of $\angle XeA$ must be equal to $1/n$ of the measure of $\angle CAY$.

Hence the problem to divide an angle into any number of equal parts depends upon the solution of the problem of drawing the respective curve.

Analysis of the method of drawing a curve as the geometrical locus of vertices of triangles where the remote angle at the base is equal to $(n-1)$ times the angle at the vertex.

Let XY be a given straight line from which (in relation to the points X and A) a n-sectorial curve (a) is drawn. In Fig. 2, $n=6$. Connect any point a_6 of the curve with the relative points X and A.

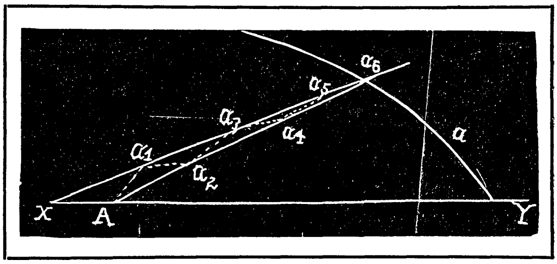


Fig. 2.

In $\triangle a_6XA$, $\angle a_6XA= 5(\angle Xa_6A)$, by hypothesis. Take point A as a center and with AX as a radius circumscribe a circle which intersects line a_6X in a_1. In $\triangle Aa_1a_6$, the exterior $\angle Xa_1A$ is equal to $\angle a_1Aa_6+\angle Aa_6a_1$; but $\angle Xa_1A=\angle a_1XA$ by construction, and $\angle a_6XA=5(\angle Xa_6A)$ by hypothesis. Hence $5(\angle Xa_6A)=\angle a_1Aa_6+\angle Xa_6A$, or $\angle a_1Aa_6=4(\angle Xa_6A)$. For general n, $\measuredangle a_nXA=(n-1)(\measuredangle Xa_nA)$, $\measuredangle a_1Aa_n=(n-2)(\measuredangle Xa_nA)$.

Take point a_1 as a center and with $a_1A=AX$ as a radius circumscribe a circle which will intersect line AC at a_2; then, reasoning as above, we will find that $\angle a_na_1a_2=(n-3)(\angle Xa_nA)$. Repeating this construction, we will form new triangles, where one angle at the base will equal respectively $(n-4)$, $(n-5)$, ..., 4, 3, 2, and 1 times $\angle Xa_nA$. The last triangle will also be an isosceles triangle. Therefore, we must consider $\triangle Xa_6A$ as a sum of a series of isosceles triangles, where all sides are equal to XA.

The nearer to XY point a_6 on the curve is taken, the less is the difference between XY and Xa_6, the less the difference between AY and Aa_6; and when point a_6 coincides with Y, the broken line $Aa_1a_2...a_6$ straightens itself and coincides with AY. Hence the relative points A and X for a required curve are: For a bisectorial curve, $XA=AY$; for a trisectorial curve, $XA=\frac{AY}{2}$; ...; for a n-sectorial curve,

$$XA=\frac{AY}{n-1}.$$

It remains to construct the instrument for drawing the curves.

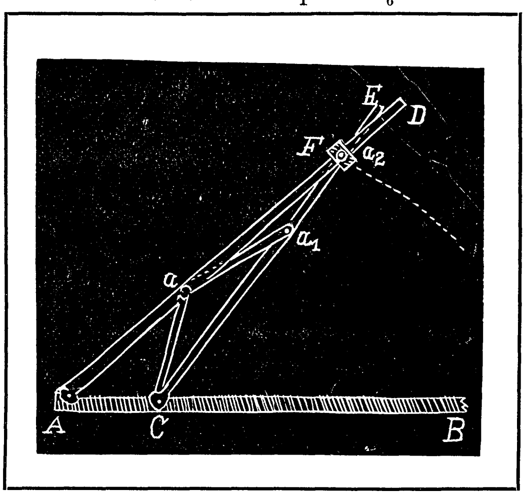


Fig. 3.

This instrument (Fig. 3) consists of a ruler AB supplied with pins so that when ruler AB is placed in coin-

cidence with any straight line it will become fixed. At the points A and C the rulers AD and CE are attached in hinge-like fashion so that they may be freely movable. These rulers are also supplied with slits along their whole length. Also, at the point C hinges a folded ruler $Caa_1a_2a_3...$ consisting of parts Ca, aa_1, ..., each part being equal to AC. The rulers AD, CE, and the folded ruler $Caa_1a_2...$ are held together by a common handle F, which may be removed and replaced. This handle is supplied with a pencil for drawing.

When the ruler AB is placed on a straight line and the instrument is folded, the straight line is visible through the open slits of rulers CE and AD.

With this instrument we are able to solve the problem:

Draw a curve as the geometrical locus of vertices of triangles, the remote angle at the common base of which is $(n-1)$ times the angle at the vertex.

Solution: Apply instrument AB (Fig. 4) to the line XY. Bring down the ruler CE until its slit coincides with line XY. Unfold the folded ruler $Caa_1a_2a_3$ ... and take $(n-1)$ parts of it. Connect the odd points of the folded ruler with the ruler CE. Bring down the ruler AD until its slit coincides with line XY, and connect the even points of the folded ruler with the ruler AD. Now apply the handle F at the $(n-1)$st part of the folded ruler, fixing the rulers CE and AD. Move handle F (supplied with a pencil). The instrument will draw the required curve.

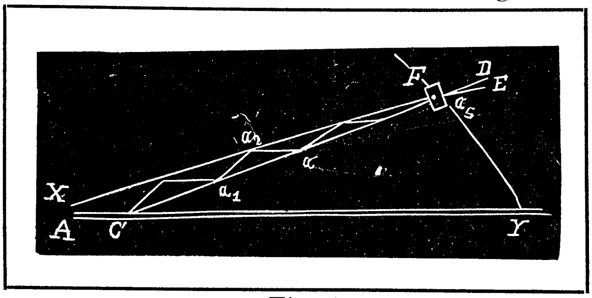


Fig. 4.

Let us prove that the drawn curve is the one required.

We have to prove that $\angle FCY$ is divisible geometrically into n equal parts.

Proof: $\triangle XFC$ is the sum of a series of $(n-1)$ isosceles triangles where the angles at the bases are successively equal to 1, 2, 3, 4, ..., $(n-1)$ times the angle at the vertex of $\triangle XFC$, by the construction of the instrument. [The points a, a_2, a_2, ..., of the folded ruler $Caa_1a_2a_3...$ will slide between the directrices AD and CE.] Therefore, in $\triangle XFC$, $\angle FXC=(n-1)(\angle XFC)$. Hence $\angle FCY$ is divisible into n equal parts.

But the triangles formed by moving handle F of the two connected directrices will include the series of $(n-1)$ isosceles triangles. Therefore the formed curve is the required one.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

269. Proposed by C. N. SCHMALL, College of the City of New York.

Two ferry-boats started simultaneously from opposite sides of a river and one being faster than the other, they met 720 yards from the shore. Each boat remained 10 minutes in its slip to change passengers and started on its return trip, when it was found that they met again 400 yards from the other shore. What is the width of the river?

Solution by THEODORE L. De LAND, Treasury Department, Washington, D. C.

Let x=the width of the river in yards; y=the speed of the slower boat in yards per minute, and z=the speed of the faster boat in yards per minute.

Then $720\div y$=the time for the slower boat to travel 720 yards, and $(x-720)\div z$=the time for the faster boat to travel $x-720$ yards; and as the two boats now meet, we equate the time, and have after reduction:

$$720z=(x-720)y\ldots(1).$$

Then $(x-720)\div y+10+400\div y$=the time for the slower boat to reach the shore, wait ten minutes for passengers, and travel 400 yards; and $720\div z+10+(x-400)\div z$=the time for the faster boat to reach the shore, wait ten minutes for passengers, and travel $x-400$ yards; and as the two boats now meet again, we equate the time as before, and have after reduction:

$$(x+320)y=(x-320)z\ldots(2).$$

Multiply (1) by (2), member by member, to eliminate y and z, and we have:

$$720(x+320)=(x-720)(x-320)\ldots(3).$$

From (3) we find $x=0$ or 1760 yards=1 mile, the width of the river.

From (1), after substituting for x, we have, $y:z::9:13$; or the speed of the slower boat is to the speed of the faster boat as 9 is to 13.

Also solved by L. E. Newcomb, Daniel B. Northrop, A. H. Holmes, J. E. Sanders, G. W. Greenwood, Frank M. Dyzer, and L. H. Rice.

270. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics in The University of Pennsylvania, Philadelphia, Pa.

Find the simplest integral form of the sum $y(y-1)\ldots(y-x)+2y(2y-1)\ldots(2y-x)+\ldots+zy(zy-1)\ldots(zy-x)$.

No solution has been received.

271. Proposed by L. E. NEWCOMB, Los Gatos, California.

Sum the series $\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+\ldots$ to ∞, $b>a$.

I. Solution by A. H. HOLMES, Brunswick, Maine.

Let $S=\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+$ etc., to infinity, $b>a$.

Put $\frac{a}{b}=c$, then $S=c+\frac{c^3}{3}+\frac{c^5}{5}+$ etc. $\therefore \frac{dS}{dc}=1+c^2+c^4+$ etc.

Let $c^2=e$. Then $\frac{2cdS}{de}=1+e+e^2+e^3+$ etc. $=\frac{1}{1-e}$, $dS=\frac{de}{2c(1-e)}=\frac{dc}{1-c^2}$.

$\therefore S=\frac{1}{2}\log\left(\frac{1+c}{1-c}\right)=\frac{1}{2}\log\left(\frac{b+a}{b-a}\right)$.

II. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Subtracting $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots$ from $\log(1+x)=+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$, we get $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots=\frac{1}{2}\log\frac{1+x}{1-x}$.

Substituting $x=\frac{a}{b}$, we obtain $\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+\ldots=\frac{1}{2}\log\frac{b+a}{b-a}$.

GEOMETRY.

294. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $(x^2+y^2)^3=mx^3$ to the problem of finding a cube m times a given cube.

I. Solution by the PROPOSER.

Construct by points the locus of the equation. Take a value of x equal to the side of the given cube. Join the corresponding point of the curve to the origin. Denote this line by a. Construct the line numerically equal to the square root of a. Then evidently the cube of this line is mx^3; or this line is the side of a cube m times as large as the given cube.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

On $AB=\sqrt[3]{m}$ describe a circle. On AB lay off $AQ=x$, the edge of the given cube. At Q, erect a perpendicular intersecting the circle at P. Let $PQ=y$. Then since $AP^2=\sqrt[3]{(m)}AQ$, we have $x^2+y^2=\sqrt[3]{(m)}x$, and, therefore. $(x^2+y^2)^3=mx^3$. Hence, the cube on AP is always m times the cube on AQ.

296 (Incorrectly numbered 294). Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

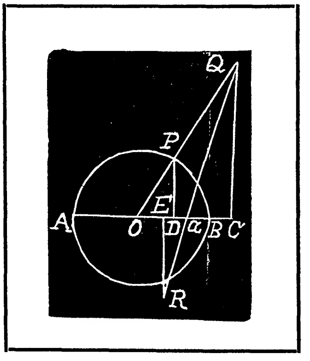
a) Suppose an indefinite line be pivoted at the end of a revolving radius whose center is the origin; and the initial position of the radius is coincident with the X-axis and the pivoted line perpendicular to it. As the radius revolves through equal amounts of arc the line moves to the right over corresponding equal intercepts on the X-axis. What is the equation of the locus of a point on the line whose distance from the end of the radius is equal to a diameter?

b) Show how the locus can be applied to the multisection of an angle.

c) Suppose the diameter be laid off in both directions.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let AB be a diameter of the given circle; $\angle POB=\theta=$angle OP makes with AB; a, the intersection of the line with AB; $PQ=PR=AB=2r$; $Ba=r\theta/90$; $Oa=r(1-\frac{1}{90}\theta)$; $Pa=r\sqrt{[1+(1-\frac{1}{90}\theta)^2-2(1-\frac{1}{90}\theta)\cos\theta]}=A$; $aQ=2r+A$; $aR=2r-A$; $PD=r\sin\theta$; $OD=r\cos\theta$.



$$QC=\frac{DP.Qa}{Pa}=\frac{r\sin\theta(2r+A)}{A}=y, \text{ for } Q \text{ locus.}$$

$$ER=\frac{r\sin\theta(2r-A)}{A}=y, \text{ for } R \text{ locus.}$$

$$aC=\frac{r}{A}(2r+A)(\cos\theta-1+\frac{\theta}{90}).$$

$$aE=\frac{r}{A}(2r-A)(\cos\theta-1+\frac{\theta}{90}).$$

$$OC=\frac{r(2r+A)\cos\theta-2r^2(1-\frac{1}{90}\theta)}{A}=x, \text{for} Q \text{locus.}$$

$$OE=\frac{2r^2(1-\frac{1}{90}\theta)-r(2r-A)\cos\theta}{A}=x, \text{ for } R \text{ locus.}$$

$$\therefore \rho^2=x^2+y^2=5r^2\pm\frac{4r^3-4r^3\cos\theta(1-\frac{1}{90}\theta)}{A}.$$

The plus sign is for Q, the minus for R. The same curve will be generated in the opposite way for the diameter (part) to the left of O.

Let ϕ be the angle to be multisected. Then $r\phi/90$ is the part of BO cut off by the pivoted line. Let $\theta=\phi/m$; this value of θ gives Pa. Let $Ba=r\phi/90m$. With a as center and Pa as radius, intersect the circle at P. Draw OP. Then POB is the required angle, ϕ/m. So on for any section or any number of sections.

299. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Show that the circle on any focal radius of an ellipse touches the auxiliary circle.

Solution by B. F. FINKEL. Ph. D., Drury College, Springfield, Mo.

Taking the left hand focus of ellipse as origin of rectangular coördinates, the axis of abscissas coinciding with the major axis of the ellipse, we have for the equation of the ellipse, auxiliar circle, and radius vector, respectively

$$\frac{(x-ae)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1\ldots(1);\quad (x-ae)^2+y^2=a^3\ldots(2);\quad y=mx\ldots(3).$$

Solving (1) and (3), we find $x=\dfrac{a(1-e^2)[e\pm\sqrt{(1+m^2)}]}{1-e^2+m^2}=k$, say. Then $y=mk$. Hence, the coördinates of the center of the circle described on the focal radius are $\frac{1}{2}k$, $\frac{1}{2}mk$, and the equation of this circle is

$$(x-\tfrac{1}{2}k)^2+(y-\tfrac{1}{2}mk)^2=\tfrac{1}{4}k^2(1+m^2)\ldots(4).$$

Solving (2) and (4), we find that the discriminant of the resulting quadratic equations vanish, showing that the roots are equal. Hence, (4) intersects (2) in two coincident points and is, therefore, tangent to it.

CALCULUS.

222 (Incorrectly numbered 221, p. 117). Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, O.

If a, b, c, ... represent all the prime numbers 2, 3, 5, ... prove that

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\ldots\ldots=\frac{15}{\pi^2}.$$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\ldots$$

$$=\frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots}=\frac{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots}{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots}$$

$$=\frac{\pi^5/6}{\pi^4/90}=\frac{15}{\pi^2}.$$

(See Vol. V, No. 5, May, 1898, page 134, third line from bottom, together with what precedes, for the complete solution. Also see Vol. XIII, No. 2, February, 1906, pages 40—41.)

223. Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Prove that $\displaystyle \underset{n\doteq\infty}{\text{Lim.}}\ \frac{\sum\limits_{\lambda=1}^{n}\lambda^k}{n^{k+1}}=\frac{1}{k+1}.$

I. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Putting $h=\frac{1}{n}$, we get $\underset{\substack{h\doteq 0\\ nh=0}}{\text{Lim.}}\ h[h^k+(2h)^k+(3h)^k+\ldots+(nh)^k]$

$$=\int_0^1 x^k dx=\frac{1}{k+1}.$$

II. Solution by EDWIN L. RICH, Schenectady, N. Y.

Here $\sum\limits_{\lambda=1}^{n}\lambda^k=\frac{n^{k+1}-1}{k+1}$, treating the summation as an integration. Then we have $\underset{n\doteq\infty}{\text{Limit}}\left(\frac{1}{(k+1)}\right)\left(\frac{n^{k+1}-1}{n^{k+1}}\right)$. Since this is in the form $\frac{\infty}{\infty}$, differentiate the numerator and denominator $k+1$ times, dividing numerator and denominator by $(k+1)$, (k), $(k-1)$, ..., $(k-k+1)$ successively, and we have

$$\underset{n\doteq\infty}{\text{Lim.}}\ \frac{\sum\limits_{\lambda=1}^{n}\lambda^k}{n^{k+1}}=\frac{1}{k+1}.$$

We might arrive at the same conclusion, by considering n finite, then $\frac{1}{k+1}\left(\frac{n^{k+1}-1}{n^{k+1}}\right)=\frac{1}{k+1}\left(1-\frac{1}{n^{k+1}}\right)$, which when $n\doteq\infty$, approaches $\frac{1}{k+1}$ as a limit.

III. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

One of the most general formulae for the summation of series is that of Euler, who has stamped his genius upon so many branches of mathematics. It is

$$\Sigma u=\int u\partial x+\tfrac{1}{2}u+\frac{B_1}{1.2}\cdot\frac{\partial u}{\partial x}-\frac{B_3}{4\,!}\frac{\partial^3 u}{\partial x^3}+\frac{B_5}{6\,!}\cdot\frac{\partial^5 u}{\partial x^5}-\ldots+\ldots$$

where B_1, B_3, B_5, denote the famous Bernoulli's Numbers $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc. Putting $u=x^k$, we have

$$\Sigma x^k=\frac{x^{k+1}}{k+1}+\tfrac{1}{2}x^k+\frac{B_1}{2}\cdot\left(\frac{n}{1}\right)x^{k-1}-\ldots$$

Consequently we have

$$\frac{\Sigma n^k}{n^{k+1}}=\frac{n^{k+1}}{n^{k+1}}\left(\frac{1}{k+1}+\tfrac{1}{2}\cdot\frac{1}{n}+\tfrac{1}{6}\cdot\tfrac{1}{2}\cdot\frac{1}{n^2}+\ldots\right)=\frac{1}{k+1}+\tfrac{1}{2}\cdot\frac{1}{n}+\ldots$$

For $n=\infty$, all the terms save the first cancel, and we have $\frac{1}{k+1}$ as the limiting value. The sum of the kth powers of the natural series of numbers can also be found without the aid of calculus by the theory of undetermined coefficients.

227. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Prove that $\int_0^\infty \tan^{-1}(\tan\alpha\,\sin x)\frac{dx}{x}=\frac{1}{2}\pi\log(\tan\alpha+\sec\alpha)$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $\tan\alpha=m$. $\int_0^\infty \tan^{-1}(\tan\alpha\,\sin x)\frac{dx}{x}=\int_0^\infty \tan^{-1}(m\sin x)\frac{dx}{x}$.

$$\tan^{-1}(m\sin x)=m\sin x-\tfrac{1}{3}m^3\sin^3x+\tfrac{1}{5}m^5\sin^5x-\tfrac{1}{7}m^7\sin^7x+\dots$$
$$\sin^3x=-\tfrac{1}{4}(\sin3x-3\sin x)$$
$$\sin^5x=\tfrac{1}{16}(\sin5x-5\sin3x+10\sin x)$$
$$\sin^7x=\tfrac{1}{64}(\sin7x-7\sin5x+21\sin3x-35\sin x)$$

. . . .

$$\int_0^\infty\frac{\sin x\,dx}{x}=\frac{\pi}{2},\quad \int_0^\infty\frac{\sin^3x\,dx}{x}=\frac{\pi}{4},$$
$$\int_0^\infty\frac{\sin^5x\,dx}{x}=\frac{3\pi}{16},\quad \int_0^\infty\frac{\sin^7x\,dx}{x}=\frac{5\pi}{32}.$$

$$\therefore\int_0^\infty\tan^{-1}(m\sin x)\frac{dx}{x}=\frac{\pi}{2}(m-\tfrac{1}{6}m^3+\tfrac{3}{40}m^5-\tfrac{5}{112}m^7+\dots)$$
$$=\frac{\pi}{2}\log[m+\sqrt{(1+m^2)}].$$

$$\therefore\int_0^\infty\tan^{-1}(\tan\alpha\,\sin x)\frac{dx}{x}=\frac{\pi}{2}\log(\tan\alpha+\sec\alpha).$$

II. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

According to Frullani's Theorem (see Williamson's *Integral Calculus*, 119, page 155), we have

$$\int_0^\infty\frac{\tan^{-1}ax-\tan^{-1}bx}{x}dx=\frac{\pi}{2}\log\frac{a}{b}.$$

Putting in the proposed integral $\tan\alpha=m$, we have

$$\int_0^\infty\tan^{-1}(m\sin x)\frac{dx}{x}$$

$$=\int_0^\infty\{\tan^{-1}[\sqrt{(1+m^2)}+m]\sin x-\tan^{-1}[\sqrt{(1+m^2)}-m]\sin\}\frac{dx}{x}$$

$$=\frac{\pi}{2}\log\frac{\sqrt{(1+m^2)}+m}{\sqrt{(1+m^2)}-m}.$$

Substituting $\tan\alpha$ for m, we get

$$\frac{\pi}{2}\log\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{\pi}{2}\log\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}=\frac{\pi}{2}\log\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2=\pi\log(\tan\alpha+\sec\alpha),$$

which result is quite as great as that given in the proposed problem.

MECHANICS.

188. Proposed by H. L. ORCHARD, M. A., B. Sc.

Spherical bubbles are rising in water. Find the relation between radius and velocity.

No solution has been received.

189 (Incorrectly numbered 190). Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Give the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905).

No solution has been received.

190 (Incorrectly numbered 191). Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A pole hinged at the bottom leans against the mid-point of a smooth rope suspended from two supports of equal height. Determine the position of equilibrium.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Assuming that the figure is symmetrical with respect to the plane bisecting perpendicularly the segment determined by the two points of support, the tensions of the portions of the rope resolved parallel to the pole are equal in magnitude and direction, and must therefore be zero since the rope is smooth. Hence, the pole rests in a position perpendicular to the plane determined by the portions of the rope.

191 (Incorrectly numbered 192). Proposed by REV. J. H. MEYER, S. J., College of Sacred Heart, Augusta, Ga.

Find the velocity of a planet at a given point in its orbit.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

In the case of any central orbit we have $v=h/p$...(1), where p is the perpendicular from the pole upon the tangent at the point, and h is a constant. Also,

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$$

and the equation of an ellipse, referred to a focus, is $lu=1+e\cos\theta$, where l is the semi-latus rectum.

$$\therefore \frac{1}{p^2}=\frac{2(1+e\cos\theta)-(1-e^2)}{l^2}=\frac{2u}{l}-\frac{1}{al}.$$

Hence (1) may be written $v^2=\frac{h^2}{l}\left(\frac{2}{r}-\frac{1}{a}\right)$.

Also solved by the Proposer.

DIOPHANTINE ANALYSIS.

132. Proposed by DR. OSWALD WEBLEN, Princeton University, Princeton, N. J.

From the numbers, 0, 1, 2, ..., 42, select seven, such that the 42 differences of these seven numbers shall be congruent (mod. 43) to the numbers 0, 1, 2, ..., 42. The differences may be both positive and negative.

Discussion by F. H. SAFFORD, Ph. D., University of Pennsylvania.

For convenience, let a circle be divided at 43 points, numbered consecutively from 0 to 42. Take any seven of these *numbers* and call the number of spaces between adjacent numbers *intervals*. The 42 differences of these seven numbers are obtained by taking the seven intervals together with all possible sums of *adjacent* intervals, two, three, ..., six, at a time. All differences may be considered positive by this method.

In order that the seven *numbers* may be a solution it is necessary and sufficient that the *intervals* shall possess the following properties:

(1) They shall be so selected that their sum is 43.

(2) They shall be capable of being arranged so that their sums, taken one at a time, two adjacent, three adjacent, etc., shall all be different. It will be evident that no interval can be 0, otherwise the differences will not be all distinct. Also 1 and 2 must be intervals, while if 3 is not an interval 4 must be, in order to provide for the differences 1, 2, 3, 4.

It is not difficult to write the totality of selections for (1), and there are 77 of them.

In permuting each of the 77 sets, it will be found by trial that no set will satisfy (2). Hence the problem is impossible.*

AVERAGE AND PROBABILITY.

160. Proposed by J. F. LAWRENCE, A.B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Okla.

Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions. Find the mean value of that portion containing the center of gravity.

No solution has been received.

*There are solutions of the corresponding problem in which 43 is replaced by 3, 7, 13, or 21. See proposed problem 142. Ed. D.

163. Proposed by R. D. CARMICHAEL, Anniston, Ala.

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle?

No solution has been received.

164. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Tex.

If m is prime, and the numbers 0, 1, 2, ..., m^2-1 are placed at random in the form of a square, the probability that the square is hyper-magic is $(m-1)m/(m^2-2)!$

No solution has been received.

167. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

Solution by the PROPOSER.

Let a, b, c, ... be the p segments in some particular case. The chance that another point taken at random shall fall on the first of these segments is a/l; on the second, b/l; and so on. Hence the chance in this particular case that p new points taken at random will fall each on one of the p segments a, b, c, ... in an assigned order is $abc.../l^p$. The chance that they shall so fall in any order is evidently $p!abc.../l^p$. Hence the probability of this occurring, however the line is divided and however the p points are chosen, is $p!m(abc...)/l^p$.

If now we can find another expression involving no unknown quantity and giving this same probability, the two will enable us to determine the value of $m(abc...)$.

The number of ways in which p points can be taken one on each of p chosen segments is the same as the number of ways in which $p-1$ points can be placed one between the two of each consecutive pair of p other points, and is easily found by the theory of permutations to be $p!(p-1)!$. But p segments may be chosen from n segments in $\dfrac{n!}{p!(n-p)!}$ ways. Hence the whole number of ways in which p points can be chosen on n segments one on each of p segments is

$$\frac{p!(p-1)!\,n!}{p!(n-p)!}=\frac{n!(p-1)!}{(n-p)!}.$$

Now the whole number of ways in which $n-1+p$ points can be arranged is $(n+p-1)!$. Hence the chance that no two of the p points last chosen shall be on the same one of the original n segments is

$$\frac{n!(p-1)!}{(n-p)!}\div(n+p-1)!=\frac{n!(p-1)!}{(n-p)!(n+p-1)!}.$$

But this is the same chance as that which was found above in different terms. Equating the two, we have

$$\frac{p!\,m(abc...)}{l^p}=\frac{n!(p-1)!}{(n-p)!(n+p-1)!}.$$

Hence, $m(abc...)=\dfrac{l^p.n!(p-1)!}{p!(n-p)!(n+p-1)!}$.

For the case $p=n$ this value reduces to $\dfrac{l^n(n-1)!}{(2n-1)!}$, the mean value of the product of the n segments of the line. This result agrees with that found by Mr. Crofton in the solution of the latter problem in the Encyclopedia Britannica, Vol. XIX, p. 784.

This solution should have appeared instead of the solution published last month. Both solutions were received before the last issue went to press. By an oversight the defective solution got in with the material for publication. Ed. F.

170. Proposed by LON C. WALKER, Santa Barbara, Cal.

Find the area of a triangle formed by drawing a line at random through each of three points taken at random within a given triangle.

No solution has been received.

174. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A chord of length C is drawn at random in a given ellipse. What is the average area of the segment cut off by the chord?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Refer the ellipse to conjugate diameters so that $x^2/m^2+y^2/n^2=1$ is its equation.

$$\text{Area of segment}=\frac{2n}{m}\sin\omega\int_{(m/2n)\sqrt{(4n^2-c^2)}}^{m}\sqrt{(m^2-x^2)}dx$$

$$=mn\sin\omega\left[\cos^{-1}\left(\frac{\sqrt{(4n^2-c^2)}}{2n}\right)-\frac{c\sqrt{(4n^2-c^2)}}{4n^2}\right]$$

$$=ab\left[\cos^{-1}\left(\frac{\sqrt{(4n^2-c^2)}}{2n}\right)-\frac{c\sqrt{(4n^2-c^2)}}{4n^2}\right]=A.$$

ω is the inclination of m, n.

$$\text{Average area is } \triangle=\frac{\int_b^a A\,dn}{\int_b^a dn}.$$

$$\therefore \triangle = \frac{ab}{a-b}\left[a \cos^{-1}\left(\frac{\sqrt{(4a^2-c^2)}}{2a}\right) - b\cos^{-1}\left(\frac{\sqrt{(4b^2-c^2)}}{2b}\right)\right.$$

$$\left. + \frac{c}{4}\left(\frac{\sqrt{(4a^2-c^2)}}{a} - \frac{\sqrt{(4b^2-c^2)}}{b}\right)\right].$$

Corollary. If $a=b$, $\triangle = a^2\left[\cos^{-1}\left(\frac{\sqrt{(4a^2-c^2)}}{2a}\right) - \frac{c\sqrt{(4a^2-c^2)}}{4a^2}\right]$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

275. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given the simultaneous equations $x^y - y^x = 0$ and $y - x = a(a+1)^{1/a}$; find a solution which is real when $a > -1$.

276. Proposed by W. J. GREENSTREET, M. A.. Editor of The Mathematical Gazette, Stroud, England.

If $x_1, x_2, \ldots, x_n$ be unequal and $f(n)$ be a rational integral function of degree $\not> n-2$, then shall

$$\sum_{r=n-1}^{r=1} \frac{f(x_r)}{(x_r-x_1)(x_r-x_2)\ldots(x_r-x_n)} = 0.$$

277. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

If $\alpha, \beta, \gamma, \delta$ are the roots of the quartic $ax^4+bx^3+cx^2+dx+e=0$, calculate the value of the product of the twelve expressions of the form $(4\alpha-2\beta-\gamma-\delta)$ in terms of H, I, J, the well known functions of the differences of the roots.

GEOMETRY.

306. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Find the length of the perpendicular let fall from the point in space $(5, 6, 7)$ upon the line $x=2z-3$, and $y=-3z+1$.

307. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

A family of planes containing a common line intersects a sphere. Find the orthogonal trajectories of the traces. An analytic solution is preferred.

308. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the locus of O, if the differences of the squares of tangents from it to circles A, B, C are x^2, y^2, z^2, respectively.

CALCULUS.

232. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin nx}{\sin x}dx$, and (b) $\int_0^{\frac{1}{2}\pi}\frac{\sin^2 nx}{\sin x}dx$, where n is a positive integer.

233. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Prove that $\int_0^\infty\frac{a^{-1}dx}{1+2x\cos\theta+x^2}=\frac{\pi\sin(1-a)\theta}{\sin a\pi\sin\theta}$

MECHANICS.

196. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From a uniform, solid right circular cone two planes cut off a portion such that the sections are similar ellipses with co-planar axes (not parallel). The centers of the elliptic faces are O_1, O_2, and the center of gravity of the solid is G. GX parallel to O_1O_2 cuts the axis of the cone in X. Find GX/O_1O_2 in terms of the ratio of the major axes of the ellipses.

197. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

Suppose that a primary planet and its satellite revolve with uniform angular velocity in circular orbits in the same plane. What relation must hold between the radii of their orbits and their angular velocities in order that the curve traced by the satellite shall be everywhere concave to the sun? Apply to the earth-moon system to prove that the moon's path is always concave to the sun.

DIOPHANTINE ANALYSIS.

142. Proposed by DR. L. E. DICKSON, The University of Chicago.*

Let n be an integer >1 and set $p=n(n-1)+1$. Required n integers whose $n(n-1)$ differences are congruent (modulo p) to the numbers 1, 2, ..., $p-1$. Exhibit at least for $n=3$, 4, 5, all inequivalent sets of solutions where a set $a_1, a_2, ..., a_n$ is called equivalent to the set $m(a_1-d), m(a_2-d), ..., m(a_n-d)$, for any integers m and d (m not divisible by p).

143. Proposed by JOHN D. WILLIAMS (being the first of his 14 challenge problems proposed in 1832).

Make $x^2+y^2=a^2=z^2+w^2$ and $x^2-w^2=z^2-y^2=\square$.

144. Proposed by JOHN D. WILLIAMS (being the ninth of his 14 challenge problems proposed in 1832).

Make $(m^2+n^2)^2x^2\pm(m^2+n^2)x=\square$, $(m^2-n^2)^2x^2\pm(m^2-n^2)x=\square$, and $4m^2n^2x^2\pm2mnx=\square$.

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*See problem 132, Diophantine Analysis, proposed by Dr. Veblen, and its discussion on page 215 by Dr. Safford.

AVERAGE AND PROBABILITY.

183. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A point within a given triangle is joined to each of the corners. What is the average of the sum of the lengths of these three lines?

184. Proposed by HENRY HEATON, Atlantic, Iowa.

Through every point of the sides of a given square, straight lines are drawn across the square in every possible direction. What is their average length?

BOOKS AND PERIODICALS.

A Short Course in Differential Equations. By Donald Francis Campbell, Ph. D., Professor of Mathematics, Armour Institute of Technology. 8vo. Cloth, vii+96 pages. New York: The Macmillan Co.

The treatment of the subject of Differential Equations in this little book is sufficiently comprehensive to give the student of Engineering a working knowledge of the subject and to enable him to handle intelligently nearly all differential equations which he is likely to encounter. At the end of each chapter is a list of well chosen problems, many of them drawn from practical affairs in Electrical and Mechanical Engineering. The book is one that will meet the wants of that class of students who have not the time nor the desire to master the more comprehensive treatises of Johnson, Boole, Forsythe, etc. B.F.F.

An Introduction to Astronomy. By Forest Ray Moulton, Ph. D., Assistant Professor of Mathematics in The University of Chicago; and Author of An Introduction to Celestial Mechanics. 8vo. Half Leather, xviii+557 pages. New York: The Macmillan Co.

In this volume, the author aims to give an introductory account of the present state of the Science of Astronomy, to present the subject in such a way that it shall be easily comprehended by students who have only a knowledge of elementary algebra and geometry and who have no extensive scientific training, and to give them a well balanced conception of the Astronomy of the present day. The author treats the subject with the same enthusiasm that characterizes his teaching, and he has told the story of Astronomy in a most interesting and fascinating way. One reads with profound interest the chapter on Evolution of the Solar System, in which is given the various inconsistencies of the Laplacian, or Nebular, Hypothesis, and in which the author advances what he calls the "Spiral Nebular Hypothesis," a theory which was first advanced by Professor Chamberlain and the author. This interesting theory explains many facts that the Laplacian Theory does not explain. The theory is supported by many facts and is being recognized by many of the leading astronomers. This book in the hands of the student with a competent instructor to lead him can not fail to arouse great interest in that most wonderful of all sciences, astronomy. B. F. F.

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MATHEMATICAL PAPERS.

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given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=v^{v/(v-1)} \text{ and } \frac{z^2}{c^2}=v^{1/(v-1)}-1.$$

This surface is symmetrical with reference to the plane $z=0$, and incloses that plane by an infinite elliptic boundary. As v increases from 0 to ∞ the sections parallel to $z=0$ decrease continually, remaining ellipses always, until they reach their limiting size, which is that of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Another companion to the same curve is

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=v^{v/(v-1)} \text{ and } \frac{y^2}{b^2}=1-v^{1/(v-1)}.$$

y has a real value only when $v^{1/(v-1)}=1$. Hence the companion degenerates to $y=0$ and $\frac{x^2}{a^2}-\frac{z^2}{c^2}=\infty$. It is therefore an hyperbola infinitely removed from the origin and lying in the plane $y=0$.

The companion to the sphere $x^2+y^2+z^2=a^2$ is

$$x^2+y^2=v^{v/(v-1)} \text{ and } z^2=a^2-v^{1/(v-1)}.$$

When $a^2=1$, the companion is an infinite circle. When $a^2<1$, the companion is imaginary. When $a^2>1$, the companion is a real surface. Every section by planes parallel to $z=0$ is a circle. For all values of v which make $v^{1/(v-1)}>a^2$ the planes which make circular sections are imaginary.

NOTE ON THE ADDITION THEOREM IN TRIGONOMETRY.

By DR. G. A. MILLER.

When $\cos\alpha$, $\cos\beta$, $\sin\alpha$, $\sin\beta$ are substituted for x_1, x_2, y_1, y_2 respectively, in the well known identity

$$(x_1^2+y_1^2)(x_2^2+y_2^2)=(x_1y_2+x_2y_1)^2+(x_1x_2-y_1y_2)^2,$$

there results

$$(\cos^2\alpha+\sin^2\alpha)(\cos^2\beta+\sin^2\beta)=(\cos\alpha\sin\beta+\cos\beta\sin\alpha)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2$$

or $$(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2=1.$$

For every value of and β the two expressions in parenthesis represent real

numbers whose squares are together equal to unity. Hence one represents $\sin\phi$ and the other $\cos\phi$, where ϕ is a function of α and β.

Suppose we know that

$$\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta=\sin(\alpha+\beta)=\sin(180^\circ-\alpha-\beta) \qquad (A).$$

It follows from the given identity that

$$\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta=\cos(\alpha+\beta) \text{ or } \cos(180^\circ-\alpha-\beta).$$

The latter is impossible since $\alpha=\beta=0$ would not satisfy the equation. Hence we have that from the given identity and (A) it follows that

$$\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta=\cos(\alpha+\beta)=\cos(-\alpha-\beta).$$

If the given identity is written in the form

$$(x_1^2+y_1^2)(x_2^2+y_2^2)=(x_2y_1-x_1y_2)^2+(x_1x_2+y_1y_2)^2$$

we have, by the method used above, that

$$\sin\alpha\,\cos\beta-\cos\alpha\,\sin\beta \text{ and } \cos\alpha\,\cos\beta+\sin\alpha\,\sin\beta$$

are respectively the sign and cosine of a given angle. It should be observed that it does not matter which of these expressions is taken for sine ϕ except that the formulas come out in a little briefer form if we proceed as above.

This note does not aim at completeness. Its object is to call the attention of teachers of trigonometry to the fact that the derivation of these formulas could proceed along lines which differ from those which are generally pursued, and that these methods offer many advantages. Other things being equal it is always desirable to exhibit connection between different formulas rather than to derive each independently.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

229. Proposed by **BENJAMIN FRANKLIN YANNEY, Mount Union College, Alliance, Ohio.**

If $a_1{}^n+a_2{}^n+a_3{}^n+\ldots+a_r{}^n=A^n$, $a_1{}^m+a_2{}^m+a_3{}^m+\ldots+a_r{}^m>$ or $<A^m$, according as $m<$ or $>n$; provided all the letters stand for positive real numbers.

Solution by the PROPOSER.

We have given $a_1{}^n+a_2{}^n+a_3{}^n+\dots+a_r{}^n=A^n\dots(1)$.

Dividing by A^n, $\left(\frac{a_1}{A}\right)^n+\left(\frac{a_2}{A}\right)^n+\left(\frac{a_3}{A}\right)^n+\dots\left(\frac{a_r}{A}\right)^n=1\dots(2)$.

Since the sum of the terms in the left member of (2) is 1, each term is <1. Hence each of the fractions

$$\frac{a_1}{A},\quad \frac{a_2}{A},\quad \frac{a_3}{A},\ \dots,\ \frac{a_r}{A}$$

is a proper fraction. Then, if in (2) we substitute m for n, we shall have

$$\left(\frac{a_1}{A}\right)^m+\left(\frac{a_2}{A}\right)^m+\left(\frac{a_3}{A}\right)^m+\dots+\left(\frac{a_r}{A}\right)^m> \text{ or } <1,$$

according as $m<$ or $>n$, as is clearly evident. Multiplying by A^m, $a_1{}^m+a_2{}^m+a_3{}^m+\dots+a_r{}^m>$ or $<A^m$, according as $m<$ or $>n$.

271. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics in The University of Pennsylvania, Philadelphia, Pa.

Find the simplest integral form of the sum $y(y-1)\dots(y-x)+2y(2y-1)\dots(2y-x)+\dots+zy(zy-1)\dots(zy-x)$.

Dr. Zerr obtains $\frac{1}{x!}\sum_{r=y}^{r=zy}\int_0^1\left[\log\left(\frac{1}{u}\right)\right]^r du$ as the sum of the series. This does not satisfy the requirements of the problem for the reason that the sum is to be in integral form. ED. F.

272. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the relations $x=\frac{ar+bs}{\lambda}=\frac{as-br}{\mu}=\frac{a\lambda-b\mu}{r}=\frac{a\mu+b\lambda}{s}$ between the finite real quantities x, a, b, r, s, λ, μ requires that $x^2=a^2+b^2$.

I. Solution by the PROPOSER.

These relations make the determinant

$$\triangle\equiv\begin{vmatrix} \lambda x+i\mu x & (ar+bs)+i(as-br) \\ (a\lambda-b\mu)+i(a\mu+b\lambda) & rx+isx \end{vmatrix},\quad (i=\sqrt{-1}),$$

necessarily $=0$; for its columns are identical. Dividing the first column by $\lambda+i\mu$ and the second by $r+is$, we have

$$\triangle\equiv(\lambda+i\mu)(r+is)\begin{vmatrix} x & a-ib \\ a+ib & x \end{vmatrix}=0.$$

Hence,

$$\begin{vmatrix} x & a-ib \\ a+ib & x \end{vmatrix}=0;\ \text{or } x^2=a^2+b^2.$$

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and A. H. HOLMES, Brunswick, Me.

By clearing of fractions, adding the four resulting equations and solving for $x-a$, we have

$$x-a=b(\lambda-\mu-\nu+s)/(\lambda+\mu+\nu+s)\dots(1).$$

Also, after the equations are cleared of fractions, by subtracting the sum of the second and third from the sum of the first and fourth, and solving for $x+a$, we have

$$x+a=b(\lambda+\mu+\nu+s)/(\lambda-\nu-\mu+s)\dots(2).$$

Multiplying (1) by (2), we have $x^2-a^2=b^2$ or $x^2=a^2+b^2$.

Also solved by G. W. Greenwood and G. B. M. Zerr.

Professor Greenwood furnished two solutions, the first of which made use of the vanishing of the determinant of the coefficients of λ, μ, ν, s in order that the four equations be consistent. The second solution was obtained by finding the values of μ/ν, ν/λ, s/λ in any three of the four equations and substituting them in the remaining equation.

Dr. Zerr solved each of the four equations for $x\lambda$, $x\mu$, $x\nu$, and xs, respectively, and added the square of the resulting equations. By dividing this last equation by $\Sigma\lambda^2$, x^2 is immediately found.

Professors Zerr and Greenwood also sent in solutions of 271.

GEOMETRY.

300. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Trisect an angle by means of a tractrix.

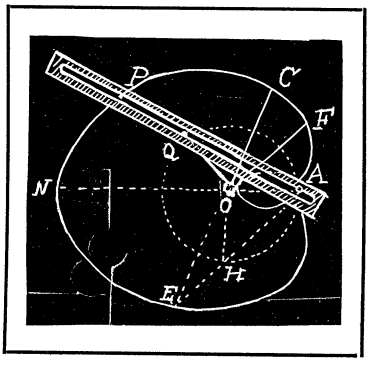
No solution has been received.

301. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $r=a(1+2\cos\theta)$ to the trisection of an angle. Describe the curve by continuous motion.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

To describe the curve by continuous motion, we may construct an instrument of the following description:



Take a bar AP and cut a path, indicated in the figure by the continuous white line, equal in length to four times the radius OA of the circle. Fasten the radius OQ at the middle point, Q, of the bar AP by means of a joint. Fasten the other end, O, of the radius at O. At P, a distance from Q equal to OQ, insert a pen point. Then if the path in the bar is set over a fixed pin at A, at a distance from O equal to OQ, the point P will describe the curve as the bar is made to move so as to keep A in the path of the bar.

To trisect an angle, for example, the angle AOC, by means of this curve, produce CO to E and draw EA. Also draw OH; then FO drawn parallel to EA makes the angle $FOC=\frac{1}{3}\angle AOC$. For since $EH=HO$, by construction of the curve, $\angle OEH=\angle EOH$. But $\angle OHA,=\angle OAH,=\angle OEH+\angle EOH=2\angle OEH$. Hence, $\angle OEH+\angle OAE,=3\angle OEH,=\angle AOC$, or $\angle OEH,=\angle FOC=\frac{1}{3}\angle AOC$.

After this department, in the last issue, was in type, we received solutions of problem 299 from Professors Scheffer, Zerr, and Greenwood. Professors Scheffer and Greenwood's solutions consisted in connecting a point, G, of the ellipse with the foci F, F'. M, the middle point of FG, is taken for the center of the circle described on the focal radius, FG, as a diameter. The line AM joining M and A, the center of the ellipse, is $\frac{1}{2}F'G$, since $AF=AF'$ and M is the middle point of FG. But $\frac{1}{2}AF'=\frac{1}{2}(2a-AF)=a-\frac{1}{2}AF$, from the definition of the ellipse. Hence, MA, the distance between the centers of the auxiliary circle and the circle described on $AF=a-\frac{1}{2}AF$, the difference of their radii. Hence the circles touch.

Dr. Zerr's solution, which was analytical, made use of the same property.

CALCULUS.

228. Proposed by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A sphere, radius r, is dropped into a conical vessel whose vertex angle is 60°. Find the contents of the vessel between the vertex and the sphere by means of the formula, $V=\int\int\int dx\,dy\,dz$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and the PROPOSER.

$x^2+y^2+z^2=r^2$ is the equation to the sphere, and $x^2+y^2=\frac{1}{3}(2r-z)^2$ is the equation to the cone. Eliminating z we get $y=\sqrt{(\frac{3}{4}r^2-x^2)}$.

$\therefore y=\sqrt{(\frac{3}{4}r^2-x^2)}=y'$ to $y=0$, $x=\frac{1}{2}r\sqrt{3}=x'$ to $x=0$.

$$\therefore\ v=4\int_0^{x'}\int_0^{y'}[2r-\sqrt{3}\sqrt{(x^2+y^2)}-\sqrt{(r^2-x^2-y^2)}]dxdy$$

$$=4\int_0^{x'}\left[r\sqrt{(\tfrac{3}{4}r^2-x^2)}-\tfrac{1}{2}(r^2-x^2)\sin^{-1}\sqrt{\frac{\frac{3}{4}r^2-x^2}{r^2-x^2}}\right.$$

$$\left.-\tfrac{1}{2}\sqrt{3}\,x^2\log\left(\frac{\sqrt{(\frac{3}{4}r^2-x^2)}+\frac{1}{2}r\sqrt{3}}{x}\right)\right]dx$$

$$=4(\tfrac{3}{16}\pi r^3-\tfrac{1}{64}\pi r^3+\tfrac{1}{12}\pi r^3-\tfrac{1}{6}\pi r^3-\tfrac{3}{64}\pi r^3)=\tfrac{1}{6}\pi r^3.$$

229. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Solve the differential equation $d^2y/dx^2=axy$.

Solution by S. A. COREY, Hiteman, Iowa, and LEROY D. WELD, Coe College, Cedar Rapids, Iowa.

Let $y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+$ etc. ...(1).

Then $d^2y/dx^2=2c_2+6c_3x+12c_4x^2+20c_5x^3+30c_6x^4+$ etc. ...(2),
and $axy=ac_0x+ac_1x^2+ac_2x^3+ac_3x^4+ac_4x^5+$ etc. ...(3).

Equating coefficients of like powers of x in (2) and (3), and reducing, we obtain by substituting in (1),

$$y=c_0\left(1+\frac{ax^3}{3!}+\frac{4a^2x^6}{6!}+\frac{4.7a^3x^9}{9!}+\frac{4.7.10a^4x^{12}}{12!}+\text{etc.}\right)$$

$$+c_1\left(x+\frac{2ax^4}{4!}+\frac{2.5a^2x^7}{7!}+\frac{2.5.8a^3x^{10}}{10!}+\frac{2.5.8.11a^4x^{13}}{13!}+\text{etc.}\right)$$

Also solved similarly by G. B. M. Zerr. Professor William Hoover did not give a solution but referred to the discussion of the general problem in Forsythe's *Differential Equations*, Chapter VII, page 217, Second Edition. A discussion of the solution of this class of differential equations, by definite integrals, is also given in Price's *Infinitesimal Calculus*, Vol. II., page 484.

MECHANICS.

187. Proposed by M. E. GRABER, M. A., Heidelberg University, Tiffin, Ohio.

Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The equations of motion are $d^2x/dt^2+\mu x=0$ and $d^2y/dt^2+\mu y=0$.

$\therefore x=A\cos t\sqrt{\mu}+B\sin t\sqrt{\mu}$, and $y=C\cos t\sqrt{\mu}+D\sin t\sqrt{\mu}$.

$\therefore (Ay-Cx)^2+(By-Dx)^2=(AD-BC)^2$, an ellipse with center of force at center.

192 Proposed by WILLIAM HOOVER, Ph D., Professor of Mathemataies and Astronomy, Ohio University, Athens, Ohio.

A solid sphere rolls down a trough formed by two planes which make with each other an angle 2α. Find, by the principle of *vis viva*, the expression for the time of rolling down the trough when the inclination of the trough to the horizon is β.

Solution by LEROY D. WELD, Coe College, Cedar Rapids, Iowa, and the PROPOSER.

Let C be the center of the sphere, O the center of the line joining the points of contact A and B of the trough and sphere, k the radius of gyration of the sphere about its center, r=its radius, θ=the angle through which a fixed radius in the plane of O and the edge of the angle 2ϕ, has rotated in any time t from the beginning of motion; let a plane and line, the first embracing AB, and the other passing through C, be drawn parallel to the edge of 2ϕ, both cutting a fixed horizontal plane, the line in the point D; x, y, the coördinates of C at the time t, D, the origin, and a and b the initial values of x, y; then by *vis viva*, m being the mass of the sphere,

$$m\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}\right)=C-2mgy.$$

When $y=b$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$, $\frac{d\theta}{dt}=0$; $\therefore C=2mgb$, and we have

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}=2g(b-y)\ldots(1).$$

Now $x=a-\theta r\sin\alpha\cos\beta$, $y=b-\theta r\sin\alpha\sin\beta$; then

$$\frac{dx}{dt}=-r\sin\alpha\cos\beta\frac{d\theta}{dt},\quad \frac{dy}{dt}=-r\sin\alpha\sin\beta\frac{d\theta}{dt},\quad \frac{dx^2+dy^2}{dt^2}=r^2\sin^2\alpha\frac{d\theta^2}{dt^2};\text{ then}$$

$$k^2\frac{d\theta^2}{dt^2}=\frac{k^2}{r^2\sin^2\alpha}\cdot\frac{dx^2+dy^2}{dt^2}.$$

Substituting in (1),

$$\frac{dx^2+dy^2}{dt^2}\cdot\frac{r^2\sin^2\alpha+k^2}{r^2\sin^2\alpha}=2g(b-y)=2g\theta r\sin\alpha\sin\beta\ldots(2).$$

But $dx^2+dy^2=ds^2$, in which $s=\theta r\sin\alpha$, or $ds=r\sin\alpha d\theta$; $\therefore$ (2) becomes

$$\frac{ds^2}{dt^2}=\frac{2r^2\sin^2\alpha}{r^2\sin^2\alpha+k^2}gs\sin\beta\ldots(3).$$

Taking the derivative of both members of (3) with respect to t, dividing by $\frac{ds}{dt}$, multiplying by dt and integrating twice, noticing that when $t=0$, $\frac{ds}{dt}=0$, and $s=0$, and finally putting $s=l$=the length of the trough,

$$s=\tfrac{1}{2}\frac{r^2\sin^2\alpha}{r^2\sin^2\alpha+k^2}gt^2,\text{ whence } t=\frac{1}{\sin\alpha}\sqrt{\frac{(10\sin^2\alpha+4)l}{5g\sin\beta}},\ k^2\text{ being }\tfrac{2}{5}r^2.$$

Also solved by G. B. M. Zerr and G. W. Greenwood.

DIOPHANTINE ANALYSIS.

136 Proposed by A H HOLMES, Brunswick, Maine.

In the equation in Diophantine Analysis: $2x^2+2x+1=\square=u^2$, show that u is always the sum of two squares.

Solution by L. E. NEWCOMB, Los Gatos, Cal.

$2x^2+2x+1=x^2+(x+1)^2\ldots(1)$.

Let $pq=x$ or $pq=x+1$ according as x is odd or even; then, for all integral values of x that satisfy (1), $\frac{1}{2}p^2-\frac{1}{2}q^2=x+1$ or $\frac{1}{2}p^2-\frac{1}{2}q^2=x$.

$\therefore p^2q^2+(\frac{1}{2}p^2-\frac{1}{2}q^2)^2=x^2+(x+1)^2=u^2$.

But $p^2q^2+(\frac{1}{2}p^2-\frac{1}{2}q^2)^2=(\frac{1}{2}p^2+\frac{1}{2}q^2)$. For p, substitute $m+n$, for q, $m-n$; then $(\frac{1}{2}p^2+\frac{1}{2}q^2)^2$ becomes $(m^2+n^2)^2$. Since $(m^2+n^2)^2=u^2$, $u=m^2+n^2$), the sum of two squares.

AVERAGE AND PROBABILITY.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A circular arc, with center at one corner of a given square, is drawn through a point taken at random in the square. What is the average length of the arc within the square?

II. Solution by HENRY HEATON, Belfield, N. D.

Let r=distance of the random point from the corner and n=number of points in the unit area. If $r<a$, the length of the arc is $\frac{1}{2}\pi r$, and the number of points from r to $r+dr$ is $\frac{1}{2}n\pi r dr$. Hence, the number of arcs of length $\frac{1}{2}\pi r$ is $\frac{1}{2}\pi n r dr$. If $r>a$ and $<a\sqrt{2}$ the length of the arc is $r[\frac{1}{2}\pi-2\cos^{-1}(a/r)]$ and the number of arcs of this length is $nr[\frac{1}{2}\pi-2\cos^{-1}(a/r)]$. To find the average, we must multiply the length of each arc by the number of times it is drawn, take the sum of the products, and divide the result by the whole number of arcs.

Hence, the required average is

$$\varDelta=\{n\int_0^a \tfrac{1}{4}\pi^2 r^2 dr+n\int_a^{a\sqrt{2}} r^2[\tfrac{1}{2}\pi-2\cos^{-1}(a/r)]^2\}\div na^2$$

$$=\frac{\pi^2}{4a^2}\int_0^{a\sqrt{2}} r^2 dr+\frac{1}{a^2}\int_a^{a\sqrt{2}} 4\{[\cos^{-1}(a/r)]^2-2\pi\cos^{-1}(a/r)\}r^2 dr$$

$$=\tfrac{2}{3}a\int_0^{\frac{1}{4}\pi}(\pi-4\theta)\sec^3\theta d\theta,\text{ where }\theta=\cos^{-1}(a/r)$$

$$=\tfrac{4}{3}a[\sqrt{2}-1+\int_0^{\frac{1}{4}\pi}\log(\tan\theta+\sec\theta)d\theta]=(1.03+)a.$$

NOTE. Since the publication of our solution of this problem in the October number of the MONTHLY, we received a criticism from Mr. Heaton, to the effect that the published solution is wrong. Mr. Sanders also agrees with Mr. Heaton in his contention. Of course we expected that criticism would follow, and do not now hope to forever settle the contention that arises when problems of this sort are proposed in the *indefinite* form. However, we again repeat what we have several times said before, that when no law of distribution is assigned there are as many ways of solving the problem as there are ways of assigning laws of distribution. In such cases, there is no such thing as *the* correct solution. The above problem is stated in the indefinite form, and so one solution assuming one law of distribution is as good as another assuming some other law of distribution. We have published Mr. Heaton's elegant solution for the benefit of those who have interpreted the problem in his way. Ed. F.

177 (Incorrectly numbered 176). Proposed by T. N. HAUN, Mohawk, Tenn.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? (Problem 72, p. 503, Williamson's *Integral Calculus*.)

A solution of this problem by Dr. Neikirk is given on pp. 180-182, Vol. VI, of MONTHLY. If any one has a different solution to offer, we shall be glad to consider it.

178. Proposed by **J. EDWARD SANDERS**, Reinersville, Ohio.

Two random planes cut a given sphere. What is the chance that they intersect within the sphere?

No solution has been received.

179. Proposed by **HENRY HEATON**, Belfield, N. D.

Through every point of the circumference of a given circle, chords are drawn in every possible direction. What is their average length?

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Parsons, W. Va., and **LEROY D. WELD**, Coe College, Cedar Rapids, Ia.

If the chord varies as the abscissa we get, since $x^2+y^2=2ax$ is the equation to the circle,

$$M=\frac{\int_0^{2a}\sqrt{(2ax)}dx}{\int_0^{2a}dx}=\tfrac{4}{3}a.$$

If the chord varies as the angle it makes with the diameter through its extremity, then, since $r=2a\cos\theta$ is the equation to the circle, we get

$$M=\frac{\int_0^{\frac{1}{2}\pi}2a\cos\theta\, d\theta}{\int_0^{\pi\frac{1}{2}}d\theta}=\frac{4a}{\pi}.$$

MISCELLANEOUS.

161 (Incorrectly numbered 157). Proposed by **H. L. ORCHARD, M. A., B. Sc.** (Unsolved problem in the Educational Times, London.)

An inelastic rod 9 feet long is placed with its upper end upon a rough vertical plane and its lower end on a smooth horizontal plane, and so that it makes an angle of 45° with each plane. It is now released, and strikes against a smooth sphere of 1 foot diameter placed in contact with the two planes. Determine the subsequent motion.

No solution has been received.

163. Proposed by **J. EDWARD SANDERS**, Reinersville, Ohio.

Two straight streams of different volumes and velocities come together. Find the path of a body floating in mid-current of either.

No solution has been received.

PROBLEMS FOR SOLUTION.

ALGEBRA.

278. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$xyz((\Sigma x)^2 < 3\Sigma y^2z\ \Sigma yz^2$, if x, y, z are positive.

GEOMETRY.

309. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

To find the equation of Brocard's Ellipse, the sides b and c of the triangle being the axes of coördinates.

310. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

Construct a plane triangle having given the base, the vertical angle, and the bisector of the vertical angle.

CALCULUS.

234. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Find the first negative pedal of an ellipse semi-axes a, b referred to origin as center, and show that its entire area is $\frac{\pi}{2}\left[\frac{(a^2+b^2)^2}{4ab}+ab\right]$.

MECHANICS.

198. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Three spheres of the same material, radii R, r, S, rest upon a horizontal plane, touching each other. Find the radius of a sphere of the same material as the others which, being placed upon the other three spheres, will just prevent the latter from separating, the coefficient of friction between the spheres being μ, and between the spheres and the table being μ'.

DIOPHANTINE ANALYSIS.

145. Proposed by JOHN D. WILLIAMS (being the twelfth of his 14 challenge problems proposed in 1832).

Make $x^2+y^2=\square$, $\frac{5}{4}(x^2+y^2)=$a cube, $xy=2x^3$, $2(x+y)+\frac{xy}{x+y}=\square$, and $(x^4+y^4)(x^2+y^2)-(x^5+y^5)\sqrt{(x^2+y^2)}=\square$.

AVERAGE AND PROBABILITY.

185. Proposed by R. D. CARMICHAEL, Anniston, Ala.

If a line l is divided into n parts by $n-1$ points taken at random on it, what is the mean value of the pth power of one of the parts taken at random?

MISCELLANEOUS.

168. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Sum to n terms, $\sin\alpha\ \sin\beta+\sin\overline{\alpha-\beta}\ \sin\overline{\beta+\gamma}+\sin\overline{\alpha-2\beta}\ \sin\overline{\beta+2\gamma}+\ldots$

NOTES AND NEWS.

Mr. George Gailey Chambers has been appointed an instructor in mathematics at the University of Pennsylvania.

J. N. Ambler, Professor of Mathematics in Roanoke College, Salem, Va., has resigned to accept a position as city engineer of Winston, N. C.

Mr. Chester E. Dimick has resigned an instructorship at the University of Pennsylvania to become an instructor in mathematics on the U. S. S. Chase, of the revenue service.

We regret to announce to our readers that, owing to increased duties at The University of Chicago and to his desire to devote all his leisure moments to investigations in a most attractive field of research in which he is now interested, Editor Dickson has decided to withdraw from the active editorship of the MONTHLY. However, he has agreed to remain with us in a semi-official way, rendering us such assistance from time to time as his leisure will permit. The work Dr. Dickson has been doing so efficiently for the past four years will now be done by Dr. H. E. Slaught, Assistant Professor of Mathematics in the University of Chicago, and Secretary of the Chicago Section of the American Mathematical Society. While we are very sorry to lose the active service of Dr. Dickson, we are glad to have his work taken up by so capable a man as Dr. Slaught. All articles for publication in the MONTHLY should in the future be sent to him instead of to Dr. Dickson.

We take this opportunity in behalf of the readers of the MONTHLY to extend to Dr. Dickson our sincere thanks for the valuable service he has rendered, through the MONTHLY, to the science of mathematics. F.